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PEARSON BACCALAUREATE

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Mathematics

2012 edition

DEVELOPED SPECIFICALLY FOR THE

IB DIPLOMA

IBRAHIM WAZIR • TIM GARRY

PETER ASHBOURNE • PAUL BARCLAY • PETER FLYNN • KEVIN FREDERICK • MIKE WAKEFORD

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Dedications

I dedicate this work to the memory of my parents.

My special thanks go to my wife Lody for standing beside me throughout writing this book. She has been my inspiration and motivation for continuing to improve my knowledge and move my career forward. She is my rock, and I dedicate this book to her.

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Tim Garry

Option 1

Chapter 1

Exercise 1.1

1 a) $P(x \geqslant 2) = 0.5248$, $P(1 \leqslant x \leqslant 3) = 0.8448$
b) $E(X) = 1.6$, $Var(X) = 0.96$
c) $E(Y) = 5.8$, $Var(Y) = 3.84$

2 a) 0.193
b) $P(12 < x \leqslant 14) = 0.743$, $P(x \geqslant 14) = 0.263$
c) $E(X) = 13.452$, $Var(X) = 2.222$
d) $E(Y) = 26.904$, $Var(Y) = 8.888$
e) $E(Z) = 26.904$, $Var(Z) = 4.444$

3 a)

x	$p(x)$		y	$p(y)$
1	0.166 667		1	0.25
2	0.166 667		2	0.25
3	0.166 667		3	0.25
4	0.166 667		4	0.25
5	0.166 667			
6	0.166 667			

b) Mean (of x) = 3.5, variance = 2.917; mean (of y) = 2.5, variance = 1.25

c)

x	$p(x)$
2	0.041 667
3	0.083 333
4	0.125
5	0.166 667
6	0.166 667
7	0.166 667
8	0.125
9	0.083 333

d) Mean = 6, variance = 4.167

4 $E(V) = 3.5$, standard deviation = 0.285

5 a) 0.1 b) 3.2 c) 1.68 d) 16 e) 21.84

6 a) $E(X + Y) = 10$, $Var(X + Y) = 3$
b) $E(X - Y) = -4$, $Var(X - Y) = 3$
c) $E(2X + 3Y) = 27$, $Var(2X + 3Y) = 17$
d) $E(2X - 3Y) = -15$, $Var(2X - 3Y) = 17$

7 a) $E(X + Y) = \sqrt{7} + \sqrt{13}$, $Var(X + Y) = 5$
b) $E(X - Y) = \sqrt{7} - \sqrt{13}$, $Var(X - Y) = 5$
c) $E(2X + 3Y) = 2\sqrt{7} + 3\sqrt{13}$, $Var(2X + 3Y) = 35$
d) $E(2X - 3Y) = 2\sqrt{7} - 3\sqrt{13}$, $Var(2X - 3Y) = 35$

8 a) $E(2X + Y) = 2\sqrt{7} + 2$, $Var(2X + Y) = 22$
b) $E(X - 3Y) = \sqrt{7} - 6$, $Var(X - 3Y) = 23$
c) $E(2X + 3Y) = 2\sqrt{7} + 6$, $Var(2X + 3Y) = 38$
d) $E(2X - 3Y) = 2\sqrt{7} - 6$, $Var(2X - 3Y) = 38$

9 a) $E(l) = 1.01$, variance = 0.0024

b)

l	$P(l)$
2.1	0.36
2	0.48
1.9	0.16

$E(l) = 2.02$, variance = 0.0048

c)

l	$P(l)$
2.85	0.064
2.95	0.288
3.05	0.432
3.15	0.216

$E(l) = 3.03$, variance = 0.0072

10 a) 0.298 b) 0.227 c) 0.298

11 0.007

12 0.560

Practice questions

1 a) 0.841
b) (i) 0.0681 (ii) 0.0312 (iii) 0.932

2 0.164

3 a) $\lambda = 3$ b) 0.647 c) 0.265
d) (i) Mean = 7, variance = 11
(ii) Not Po

4 a) 10 b) 12 c) 7 d) 35

5 a) 0.944 b) Verify

6 a) (i) Mean = 0.5, variance = 0.13 (ii) 0.0828
b) 0.904

7 a) 0.0548 b) 0.993

8 a) (i) $\frac{3}{2}$ (ii) $\frac{11}{9}$ (iii) $\frac{125}{36}$
b) 0.432

Chapter 2

Exercise 2.1

1 a) 8 b) 16 c) $\frac{5}{7}$

2 a) $\frac{1}{5}$ b) Mean = 15, standard deviation = $2\sqrt{2}$
c) $\frac{3}{5}$

3 a) $\frac{1}{10}$ b) $E(V) = 4.5$, variance $= 8.25$
c) $\frac{1}{900}$ d) $\frac{2}{5}$

4 a) 6.5 b) 11.92 c) 0.076

5 a) 4.5 b) 5.25 c) 0.078

Exercise 2.2

1 a) 0.148 b) 0.538 c) 0.686 d) 3.125

2 a) 0.007 87 b) 0.0238 c) 0.984

3 $E(N) = 125$; no; standard deviation $= 124.5$

4 a) $\frac{3}{21}$ b) $\frac{6}{21}$ c) $\frac{6}{21}$
d) 4.33 e) 2.22 f) 6

5 a) 3.3 b) 1 c) 0.657

6 a) (i) 0.7 (ii) 0.6 (iii) 4.5 (iv) 8.25
b) (i) 0.059 (ii) 1 (iii) 2.64

7 a) 0.2 b) $E(X) = 5$, $\text{Var}(X) = 20$ c) 0.328

8 a) 0.141 b) 0.316
c) $E(N) = 4$, standard deviation $= 3.464$

9 a) 0.0527 b) 0.284

10 a) 8 b) 0.573 c) 0.0648 d) 0.714

11 a) 0.128 b) 0.107

Exercise 2.3

1 a) 0.1298 b) 0.1101
c) $E(N) = 13.33$, standard deviation $= 2.108$

2 a) 0.125 b) 0.125 c) 0.0938

3 a) 0.005 67 b) 0.052 92

4 a) 0.106 b) 0.0885 c) 0.1182

5 13.33

6 0.138

7 a) 0.080 b) $E(V) = 20$, standard deviation $= 46.7$

8 a) 0.09 b) 0.0437 c) 0.991
d) (i) Mean $= 1.11$, standard deviation $= 0.351$
(ii) Mean $= 3.33$, standard deviation $= 0.609$

9 a) (i) 0.6 (ii) 0.096
b) (i) 0.360 (ii) 0.092
c) 0.173
d) $E(N) = 1.67$, standard deviation $= 1.054$
e) $E(N) = 5$, standard deviation $= 1.826$

10 a) 0.081 b) 0.0098
c) $E(N) = 30$, standard deviation $= 16.43$
d) $E(N) = 1767$, standard deviation $= 739.35$

Exercise 2.4

1 a) 0.148 b) 0.439 c) 0.899

2 $E(N) = 2.60$, standard deviation $= 0.875$

3 a) 0.288 b) 0.216 c) 0.965 d) 0.251

4 1

5 a) 0.0951 b) 0.209 c) 0.890 d) 2

6 a) 0.491 b) 0.084 c) 0.088

7 a) (i) 0.0256 (ii) 0.154
(iii) 0.662 (iv) 0.462
b) 1.87

8 a) (i) 0.420 (ii) 0.028 (iii) 0.937
b) (i) 2.14 (ii) 14.29 hours

9 8

10 a) By chance, P(at most 1 non-native) = 0.187; no reason for doubt.
b) By chance, P(at most 2 females) = 0.314; no reason for doubt.
c) $E(N) = 2.4$, standard deviation $= 1.03$
d) $E(N) = 3$, standard deviation $= 1.05$

11 a) 0.491 b) 0.150

12 a)

x	0	1	2	3
$P(X = x)$	0.399	0.461	0.132	0.0088

b) $E(X) = 0.75$, $\text{Var}(X) = 0.503$
c) 0.601

Practice questions

1 a) Answers vary b) Mean $= 33.3$, variance $= 22.2$
c) 0.0768

2 a) 0.684 b) 0.0244 c) Answers vary

Chapter 3

Exercise 3.1

1 a) $\frac{1}{8}$ b) Mean $= 2$, variance $= \frac{16}{3}$

2 a) Mean $= \frac{1+k}{2}$, standard deviation $= \sqrt{\frac{(k-1)^2}{12}} = \frac{\sqrt{3}\,|k-1|}{6}$
b) $\frac{23}{3}$

3 a) Mean $= \frac{a+b}{2}$, standard deviation $= \frac{\sqrt{3}\,|b-a|}{6}$
b) $a = 3$, $b = 9$

4 a) 0.565 b) 18.9%

5 a) 0.381 b) 0.147 c) 0.145

6 a) 0.223 b) 0.865
c) 1st quartile $= 0.575$, median $= 1.386$, 3rd quartile $= 2.773$

7 a) 0.865 b) 0.233

8 a) $E(T) = 4$ seconds, standard deviation $= 4$ seconds
b) 0.632 c) 0.320

9 0.671

10 a) 0.632 b) (i) 0.441 (ii) 0.693 c) 78 tons

11 a) 0.223
b) Median lifetime $= 13.863$, standard deviation $= 20$
c) 21.97

Chapter 4

Exercise 4.1

1 a) 0.338 b) 0.053

2 a) 0.0594 b) 0 c) 0.9973

3 a) 34.1% b) 36.65 weeks

c) Normal with $\mu = 38, \sigma = \frac{2}{\sqrt{120}}$

d) 0

e) a) and b) will change, while c) and d) will not

4 a) No

b) No, sample too small for the central limit theorem (CLT)

c) Yes, CLT applies, $p = 0.039$

5 a) 1 b) 4

6 5.06

7 a) 0.399 b) 0.154; the company's claim is fine.

8 a) 0.223 b) 0.460

c) Cannot find probability as the sample size is too small.

9 a) (i) 0.683 (ii) 0.904 (iii) 0.992

10 a) 0.001 87 b) [932.95, 987.05]

11 a) 0.837

b) No, as the sample size is too small for CLT to apply.

12 0.146

13 a) 0.009 52

b) This is so unlikely to happen. We can conclude that the claim may underestimate the true defective rate.

14 a) 0.004 40

b) This is so unlikely to happen. We can conclude that the claim may overestimate the true relief rate.

15 Approximately 0

16 22%

17 0.0548

18 a) 0.244 b) 0.271

19 0.003 35

20 a) 0.864 b) 0.941

21 1

22 a) 0.369 b) 0.004 91

23 $p \approx 7.37, \sigma \approx 1.72$

Chapter 5

Exercise 5.1

1 a) Mean = 79.333, standard deviation = 10.137

b) Mean = 0.276, standard deviation = 0.663

c) Mean = 73.067, standard deviation = 13.554

d) Mean = 47, standard deviation = 19.472

e) Mean = 66.692, standard deviation = 36.871

2 a) Mean = 499.54, standard deviation = 1.893

b) (498.50, 500.58)

c) Company's claim is acceptable

d) (498.07, 501.01)

e) 2.08, 2.94 f) (498.44, 500.64)

3 (−0.0412, 0.072 29)

4 92.5%

5 a) (995.88, 1000.1) b) 57.77%

6 a) (1002.48, 1072.52) b) 57

7 28

8 (21.1, 21.6)

9 (0.743, 2.424)

10 (0.643, 0.734)

11 (13.84, 20.28)

12 68

13 (0.1875, 0.2234)

14 1068

15 16

16 a) 0.9996 b) 3

17 (0.106, 0.425)

Practice questions

1 984

2 (2.703, 2.707)

3 a) (i) 87.03 (ii) 215.58

b) (i) (86.22, 88.04) (ii) (86.37, 87.89)

c) Greater confidence leads to less precision

4 (3.04, 4.36)

5 a) Mean = 33.18, variance = 3.22 b) (32.1, 34.2)

6 a) 96 b) 99.0%

7 a) (i) 0.45 (ii) 0.0144 (iii) (0.422, 0.478)

b) Random sampling

8 a) $(\bar{x} - 1.91, \bar{x} + 1.91)$ b) 99.0%

9 a) (11.8, 13.4) b) $\mu = 13.7$; inconsistent

10 a) (0.498, 0.557) b) 9576

11 a) 98.2% b) 10

Chapter 6

Exercise 6.1

1 There is evidence of change, p-value = 0.0339

2 There is no statistical evidence at the 1% level of significance, p-value = 1.51%

3 There is statistical evidence at the 2% level of significance, p-value = 0.274%

4 There is no statistical evidence at the 3% level of significance, p-value = 13.350%

5 There is no statistical evidence at the 5% level of significance to conclude that the wire is gold, p-value = 74.6%

6 a) There is no statistical evidence at the 5% level of significance (p-value = 38.8%) that the packs are underweight.

b) There is statistical evidence at the 5% level of significance (p-value = 2.64%) that the packs are underweight.

7 a) H_0: $p = 0.03$, H_1: $p > 0.03$. p-value = 42.7%; we do not have statistical evidence to conclude that the rate of cancer cases has increased.

b) Type II c) 73.1%

8 a) H_0: $p = 0.30$, H_1: $p > 0.30$. p-value = 0.02%; we have statistical evidence to conclude that the number of hospital stays has increased.

b) Type I. We conclude that hospital stays have increased when they actually did not.
c) 31.4%. We conclude that the number of hospital stays has not increased when it actually did.

9 a) H_0: $p = 0.54$, H_1: $p < 0.54$. p-value = 2.6%; we have statistical evidence at the 5% level of significance to conclude that consumer confidence is lower in 2009 than it was before.
b) 9.21%

10 a) H_0: $\mu = 3.2$, H_1: $\mu < 3.2$. Rejection region: $t < -1.761$, $t = -1.81$, p-value = 4.6%; we have statistical evidence to conclude that shop sales have decreased.
b) 79.7%. We conclude that the sales have not decreased when they actually did.

11 a) H_0: $\mu = 24.1$, H_1: $\mu > 24.1$. Rejection region: $t > 1.66$, $t = 1.71$, p-value = 4.5%; we have statistical evidence to conclude that the age of the consumer has increased.
b) 62.96%. We conclude that the average age has not increased when it actually did.

12 H_0: $\mu = 11.1$, H_1: $\mu > 11.1$. p-value = 0.2%; we have statistical evidence to conclude that the company's efforts are successful.

13 Matched pairs test. p-value = 2.4%; we have enough evidence that there is a difference in fuel consumption between the two car types.

14 Matched pairs test (absolute values!). p-value = 0; we conclude that the difference is more than 0.003 and hence they will not purchase the hydrostatic instruments. Type I error means that we will conclude that the difference is more than 0.003 and end up not purchasing the hydrostatic instruments; while Type II error means that we fail to see that the difference is more than 0.003 and end up purchasing the hydrostatic instruments.

15 a) Matched pairs test. p-value = 1.2%; we have statistical evidence to conclude that the passenger appears to have the worst seat.
b) 59%. We conclude that there is no difference in injury between the passenger and the driver when in fact there is a difference.

16 a) $P\left(\bar{x} > 762.34 \mid \mu = 750, \sigma = \frac{30}{\sqrt{16}}\right) < 0.05$, and hence we reject H_0.
b) p-value = 2.28%, and hence we reject H_0.
c) 15.4%

17 $\bar{x} = \frac{896}{15} = 59.73$, $s_{n-1}^2 = \frac{15}{14}\left(\frac{54\,172}{15} - \left(\frac{896}{15}\right)^2\right) = 46.50$.
H_0: $\mu = 60$, H_1: $\mu < 60$. p-value = 44%; we do not have statistical evidence to reject the company's claim.

Practice questions

1 a) 0.692 b) 320 ml c) 0.004 91
d) Enough evidence that the volume is more than 330 ml.
e) (330.43, 335.13)
f) The evidence is that the volume is not the required one.
g) (0.544, 0.896)

2 a) 0.369 b) 0.146 c) (i) 0.714 (ii) $1716.60
d) No evidence of change of standards.
e) Cannot reject the hypothesis that the data is N(68, 9).

3 a) Differences (d): 1.5, 0.6, 0.3, −0.2, 2.0, 0.6, 1.5, 0.1, 0.5, −0.4.
b) (i) H_0: $\mu_d = 0$, H_1: $\mu_d < 0$
(ii) p-value = 0.0139 > 0.01; insufficient evidence to conclude that Puzzle 2 takes longer than Puzzle 1.

4 a) H_0: $p = 0.75$, H_1: $p < 0.75$ b) p-value = 0.0530
c) (i) Reject H_0 (ii) Do not reject H_0

5 a) H_0: $\mu = 30$, H_1: $\mu \neq 30$
b) p-value = 0.114; do not reject H_0
c) t-test since population is normal and variance unknown.

6 a) H_0: $p = 0.5$, H_1: $p > 0.5$
b) (i) Critical region
(ii) Probability of finding a sample with $p \geqslant 0.733$ when the population has $p = 0.5$. The 'observed' significance level in this case is 0.0592.
c) P(Type II) = $P(X \leqslant 10 \mid p = 0.6) = 0.783$
d) (i) Type II
(ii) Conclusion will be that the coin is fair when it is not.

7 a) H_0: $\mu_d = 5$, H_1: $\mu_d < 5$ (matched pairs)
b) (i) p-value = 0.0447; cannot reject at 1% level.
(ii) Reject at 10%
c) Randomness and normality

8 Matched pairs. H_0: $\mu_d = 0$, H_1: $\mu_d \neq 0$. p-value = 0.0320; claim cannot be justified.

9 a) Critical (rejection) region
b) (i) 0.242 (ii) 0.341

10 Matched pairs. H_0: $\mu = 0$, H_1: $\mu > 0$. p-value = 0.004 09; there is enough evidence to support claim.

11 a) 0.0668 b) 9.53
c) H_0: $\mu = 75$, H_1: $\mu > 75$. p-value = 0.001 86; reject H_0.

12 a) 65
b) In both cases, H_0: $p = 0.5$, H_1: $p \neq 0.5$.
(i) Amanda: $X \sim B(3, 0.5)$;
P(Type I) = $P(X = 0 \text{ or } 3) = 0.25$
Roger: $X \sim B(8, 0.5)$;
P(Type I) = $P(X \geqslant 6 \text{ or } X \leqslant 2) = 0.289$
Amanda has the smaller Type I probability.
(ii) P(Type II) = $P(3 \leqslant X \leqslant 5 \mid p = 0.6) = 0.635$

13 a) Matched pairs. H_0: $\mu_d = 0$, H_1: $\mu_d > 0$.
b) p-value = 0.0295; we have enough evidence to conclude that practice sessions improve ability to memorize digits.

14 a) (i) $[15.0, \infty[$ (ii) $[15.8, \infty[$
b) (i) 0.440 (ii) 0.702
c) As P(Type I) decreases P(Type II) increases.

Chapter 7

Exercise 7.1

1 Frequency = 80 per day. p-value = 0.016; we can conclude that the differences are significant.

2 p-value = 0.003; we can conclude that the differences are significant.

3 p-value = 0.201; we cannot reject H_0. There is no evidence that the coins are biased.

4 p-value = 0.187; we cannot reject H_0. There is no evidence that the distribution is not binomial.

5 p-value = 0.350; we cannot reject H_0. There is no evidence that the distribution is not a Poisson distribution.

6 p-value = 0.145; we cannot reject H_0. There is no evidence that the distribution is not a normal distribution.
To modify the test, we would estimate the mean and variance from the sample and subtract two more degrees of freedom: $7 - 2 = 5$.

7

Channel	North	East	South	West
ARD	27.5	16.5	44	22
ZDF	12.5	7.5	20	10
RTL	10	6	16	8

p-value = 3.6%; we have enough evidence to reject H_0 at the 5% level of significance and conclude that there is some association between channel and region.

8 p-value = 0.281; we fail to reject H_0. We do not have evidence of any association between gender and exam classification.

9 p-value ≈ 0; we reject H_0 and conclude that we have evidence that the percentage of children taking up their parents' profession is not the same in every profession.

10 p-value ≈ 0; we reject H_0 and conclude that we have evidence that there is a relationship between the age of a user and the number of purchases he/she makes per year.

11 p-value = 0.212; we cannot reject H_0. We do not have evidence to claim that pigeons have any preference in choosing the direction of the flight after being disoriented.

12 p-value = 0.102; we cannot reject H_0. We do not have evidence to reject the claim that the data is N(5, 0.0004).

13 p-value = 0.148; we cannot reject H_0. We do not have evidence to reject the claim that the data is N(4.997, 0.000 566).

Practice questions

1 Computed $\chi^2 = 79.43 > 9.49$ and hence we reject H_0. The cost of the vehicle is not independent of the number of complaints.

2 a) $p(80.5 \leqslant X \leqslant 90.5) = 0.1455 \Rightarrow Fe = 4000 \times 0.1455 \approx 582$; others are similar.
b) Computed $\chi^2 = 53.0 > 14.07$ and hence we reject H_0. We have enough evidence to suggest that the normal distribution with mean 100 and standard deviation 10 does not fit the data well.

3 Computed $\chi^2 = 1.628 < 3.84$. We do not have enough evidence to reject H_0. Hence, we do not have enough evidence to support the claim that flu injections help reduce the number of people suffering from colds.

4 Computed $\chi^2 = 42.252 > 12.592$ and we reject the null hypothesis and conclude that we have evidence that there is some association between nicotine and alcohol consumption.

5 a) This is a t-test of the difference of two means (**Not in syllabus**). Since $t = -1.686 < 2.460$, we reject H_0. Hence, Group B gains weight faster.
b) Computed $\chi^2 = 3.469 < 5.99$ and hence we do not have enough evidence to reject the null hypothesis; therefore, there is no evidence to say that the distribution is not normal with a mean of 380.

6 a) Student's own description (reference Chapter 7)
b) 133.5, 56.3
c) $a = 9$, $b = 20$, $c = 9$
d) $\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e} = 1.0847$
e) H_0: the distribution of tree heights is normally distributed.
H_1: the distribution is not normal.
The critical number is 11.0705. Since $\chi^2 = 1.0847 < 11.0705$, we fail to reject H_0. Conclusion: we do not have enough evidence to claim that the distribution of tree heights is not normal.

7 a) χ^2 test for independence.
b) Computed $\chi^2 = 11.25 > 5.99$ and we reject the null hypothesis and conclude that we have evidence that drinking coffee has an effect on sleeping pattern.

8 Computed $\chi^2 = 0.920 < 15.507$. We do not have enough evidence to reject H_0. Hence, we do not have enough evidence of association between the day of production and the quality of the part.

9 a) (i) Computed $\chi^2 = 11.3 > 11.07$. We reject the null hypothesis and hence conclude that the die seems to be unfair.
(ii) Computed $\chi^2 = 11.3 < 15.086$. We cannot reject the null hypothesis that the die seems to be fair.
b) Student's explanation (reference Chapter 6)

10 p-value = 0.179; we cannot reject H_0 that the percentage is the same in all four professions.

11 a) Computed $\chi^2 = 7 < 16.919$ and we cannot reject the null hypothesis that the sequence contains equal numbers of each digit.
b) The probability of concluding that the sequence does not contain equal numbers of each digit when it does is 5%.

12 a) (i) 1.98 (ii) 0.33
b) p-value = 0.668; we cannot reject the hypothesis that the binomial distribution provides a good fit for the data.

13 a) H_0: there is no association between classification in exams and gender.
b)

	Distinction	Pass	Fail
Male	31.6	68.5	12.9
Female	22.4	48.5	9.12

c) 4.03
d) Degrees of freedom = 2, p-value = 0.133; we cannot reject H_0. There is insufficient evidence, at the 5% level, to conclude that there is any association between classification and gender.

14 a) $p_1 = 0.0784$, $p_2 = 0.2160$, $p_3 = 0.2960$, $p_4 = 0.280$, $p_5 = 0.1296$
b) Computed $\chi^2 = 4.8245 < 7.815$ and we cannot reject the null hypothesis.

15 a) Verify
b) n is large for the CLT to apply
c) (i) Computed $\chi^2 = 7.94 < 11.07$. We cannot reject H_0; data fit N(0, 1).
(ii) Type I error: concluding that the data do not fit N(0, 1) when in fact they do.
Type II error: concluding that data fit N(0, 1) when in fact they do not.

16 a) $f(x) > 0$, $\int_0^\infty f(x)\,dx = 1$.
b) Computed $\chi^2 = 12.0 > 7.815$. We reject the null hypothesis, i.e. f is not an appropriate model for the data.

17 a) 2.16
b) (i) H_0: Poisson law provides a suitable model.
H_1: Poisson law does not provide a suitable model.
(ii) Computed $\chi^2 = 5.35 < 13.277$. We cannot reject H_0; Poisson law may provide a suitable model.

18 a) Verify
b) Computed $\chi^2 = 1.83 < 9.488$. We cannot reject H_0; Poisson law may provide a suitable model.

19 a) H_0: distribution is B(6, 0.5); H_1: distribution is not B(6, 0.5).
p-value = 0.266; we cannot reject H_0.
b) Estimate p from the data, which would entail the loss of one degree of freedom.
c) p-value = 0.001 29; we have enough evidence to reject H_0.

20 H_0: data can be modelled by exponential distribution with mean 100 hours.
H_1: data cannot be modelled by exponential distribution with mean 100 hours.

p-value = 0.309; we cannot reject H_0. Data can be modelled by an exponential distribution with mean 100 hours.

21 a) 2.725
b) p-value = 0.0662; we cannot reject H_0. Data can be modelled by a Poisson distribution.

22 a) Mean = 1.71, variance = 0.0036
b) (i) H_0: data can be modelled by a normal distribution.
H_1: data cannot be modelled by a normal distribution.
(ii) p-value = 0.35; we cannot reject H_0. The data can be modelled by a normal distribution.

Option 2

Chapter 1

Exercise 1.1

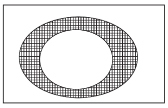
1 a) Equal b) Equal
c) Equal d) Not equal

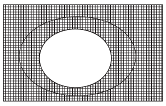
2 a) {1, 3, 4} b) {1, 3, 4} c) {6}
d) {1, 2, 5, 6} e) {6} f) {1, 2, 3}
g) {1, 2, 5}

3 a) False b) True c) True
d) True e) True f) True
g) True h) True i) True

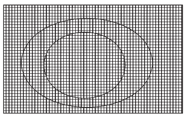
4 a) True b) True c) False
d) True e) False f) False
g) True h) False i) True

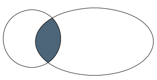
5 a) A b) B

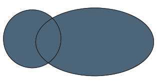
c) 

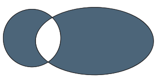
d) 

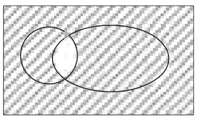
e) $\varnothing$

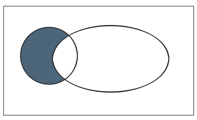
f) 

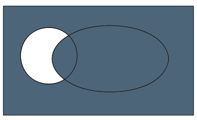
6 a) 

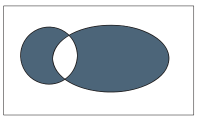
b) 

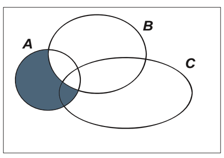
c) 

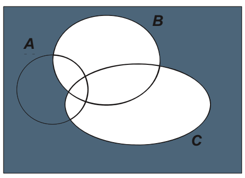
d) 

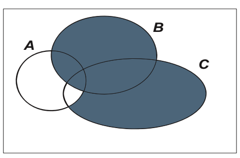
e) 

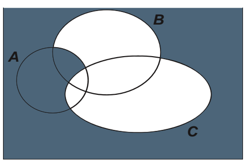
f) 

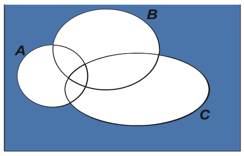
g) 

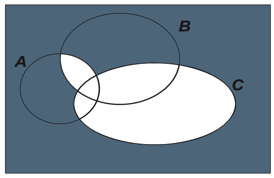
7 a) 

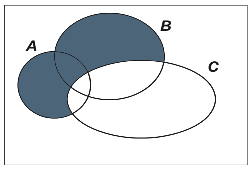
b) 

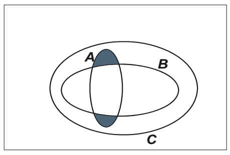
c) 

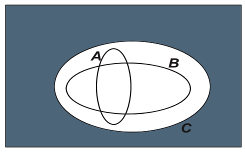
d) 

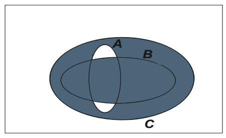
e) 

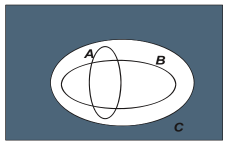
f) 

g) 

8 a) 

b) 

c) 

d-f) 

g) $\varnothing$

9 a) $\{-1\}$ b) $\varnothing$ c) $\{0, 1\}$
d) $\mathcal{P}(A) = \{\varnothing, \{0\}, \{-1\}, \{1\}, \{0, -1\}, \{0, 1\}, \{-1, 1\}, \{0, -1, 1\}\}$

10 $A \cap B'$ or $A \cap (C \setminus B)$

11 42

12 24

13 a) $\mathbb{Z}^+$ b) $\{1, 3, 5, ...\}$ c) M_6 d) $\varnothing$

14 $A = B$

15 a–l) Proof

16 128

17 a–e) Proof

18 a) Proof
b) $\{\varnothing\}$; $\{\varnothing, \{\varnothing\}\}$
c) $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$
d) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

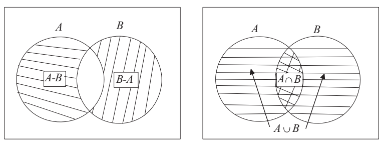
19 a) $[0, \infty[$ b) $\varnothing$ c) $[1, 3[$ d) $]0, 2]$

20 $|A \cup B| \neq |A| + |B|$

21 a–h) Proof

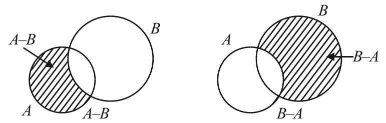
22 a–e) Proof

Practice questions

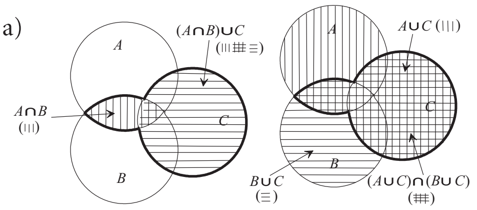
1 a) 

b) Proof

2



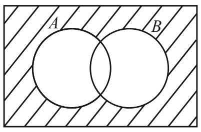
3 a)



b) Proof

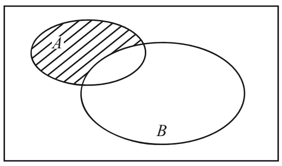
4 Proof

5 a)



b) Proof

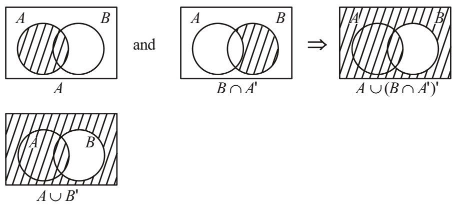
6 a)



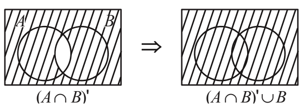
b) Proof

7 Proof

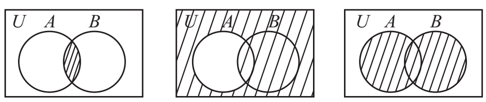
8 a)



b)



9 a)



C D E

b–c) Proof

10 Proof

11 a) (i) $S_1 = x \in \mathbb{Z}^+ \,|1 \text{ divides } x$

$= 1, 2, 3, \ldots = \mathbb{Z}^+$

(ii) $S_2 = x \in \mathbb{Z}^+ \,|2 \text{ divides } x$

$= 2, 4, 6, \ldots$

Hence, $S_2' = 1, 3, 5, \ldots.$

(iii) $S_3 = x \in \mathbb{Z}^+ \,|3 \text{ divides } x$

$= 3, 6, 9, \ldots$

Hence, $S_2 \cap S_3 = 6, 12, 18, \ldots.$

(iv) $S_6 = x \in \mathbb{Z}^+ \,|6 \text{ divides } x$

$= 6, 12, 18, \ldots$

Hence, $S_6 \setminus S_3 = S_6 \cap S_2' = \varnothing.$

b) Proof

12 Proof

Chapter 2

Exercise 2.1

1 a) (i) $\{(1, a), (1, b), \ldots, (2, c), (1, x), \ldots, (3, z)\}$
(ii) $\varnothing$
(iii) $\varnothing$
(iv) $\{(1, a), (1, b), \ldots, (2, c), (1, x), \ldots, (3, z)\}$
(v) $\varnothing$
(vi) $A \times B$
b) i and iv; ii, iii, v, and vi

2 a, c, d, e

3 a) Points on the lines $y = x$ and $y = -x$ are symmetric with respect to the x- and y-axes. For example, $(2, 2)$, $(2, -2)$, $(-2, -2)$ and $(-2, 2)$.
c) Numbers of the form n and $-n - 1$.
d) Every complete square and its positive factors.
e) Concentric circles with O as centre.

4 a) 4, 5, 4
b) 3
c) Proof

5 a) $\mathcal{R}$ is an equivalence relation. Classes are: {1}, {2}, …, {9}.
b) $\mathcal{X}$ is not an equivalence relation since it is not reflexive.

6 a) Injection b) Injection
c) Injection d) Surjection

7 a) Yes b) No c) Yes

8 a) nm b) $\dfrac{n!}{(n-m)!}$ c) $n!$

9 a) Yes; no
b) No; no
c) (i) $[-4, 3]$, $[0, 2]$
(ii) $[-9, 5]$, $[-9, 5]$, $[-1, 3]$, $[-1, 3]$
(iii) $[1, 17]$, $[1, 17]$, $[1, 5]$, $[1, 10]$

10 No; yes

11 a–b) Proof

12 a) $f(a) \neq f(b) \neq f(c)$
b) c, a, b
c) Identity; $f^{-1} = f \circ f$

13 $\mathcal{S}$ is an equivalence relation.

14 a) Proof
b) Concentric circles with centre at the origin. All points on the circle with radius $\sqrt{5}$.

15 Both. $h^{-1} : (a, b) \mapsto \left(\dfrac{2b - a}{3}, \dfrac{a + b}{3}\right)$.

16 Proof

17 $\mathcal{S}$ is an equivalence relation; $\{\{a, c, e\}, \{b, d\}, f\}$

18 {{1, 4, 6, 9, 11}, {2, 3}, {5, 10}, {7, 8}}

19 a) Not a bijection b) Bijection
c) Not a bijection

20 Proof

21 a) Proof b) {{0, 4, 8},{1, 5, 9},{2, 6},{3, 7}}
c) 3

22 a) Injective b) Not surjective

23 $f^{-1}(x, y) = \left(\dfrac{5x + 3y}{11}, \dfrac{2x - y}{11}\right)$

24 a–b) Proof

25 Proof

26 a) (i) $R = \left\{\frac{e+1}{e}, e+1\right\}$

(ii) Proof (iii) Not a surjection

b) (i) $k = \pi$ (ii) $f^{-1}(x) = \arccos(\ln(x-1))$

27 a) Proof

b) $\{\{4, 24, 32\}, \{8, 20, 36\}, \{12, 16\}, 28\}$

28 Proof

29 $h^{-1}(x, y) \mapsto \left(\frac{3y-x}{4}, \frac{x-y}{2}\right)$

30 Neither

31 a) Proof

b) $\{5k, \{1+5k, 4+5k\}, \{2+5k, 3+5k\}\}, k \in \mathbb{N}$

32 a) Proof b) $a = 2$

33 a–d) Proof

34 a–b) Proof

Practice questions

1 a) Proof

b) f is an injection.

c) S is an equivalence relation.

2 a) Proof

b) This is the set of ordered pairs (x, y) such that $x^2 + y^2 = 5$.

c) The partition is the set of all concentric circles in the plane with the origin as the centre.

3 a–b) Proof

c) $f^{-1}(x, y) = \left(\frac{2y-x}{3}, \frac{x+y}{3}\right)$

4 a) Proof

b) The classes are those pairs (a, b) and (c, d) with $\frac{a}{b} = \frac{c}{d}$.

The elements are on the same line going through the origin.

5 a) Proof

b) $\{\{a, c, e\}, \{b, d\}, \{f\}\}$

6 a) Proof

b) (i) Student explanation

(ii) $\{5, 10\}, \{1, 4, 6, 9\}, \{2, 3, 7, 8\}$

7 a) $q(x)$

b) Proof

8 a) Proof

b) $\{0, 4, 8, \ldots\}, \{1, 5, 9, \ldots\}, \{2, 6, 10, \ldots\}, \{3, 7, 11, \ldots\}$

c) 3

9 a) Proof

b) In the Argand diagram, this corresponds to the concentric circles centred at the origin.

10 a) (i–ii) f is injective but not surjective.

b) (i–ii) g is injective and surjective.

c) $g^{-1}(x, y) = \left(\frac{5x+2y}{11}, \frac{3x-y}{11}\right)$

d) Proof

11 a–c) Proof

12 a–c) Proof

13 a) $A = [e^{-1} - 1, e - 1]$

b) (i) Student explanation

(ii) Not a surjection

c) (i) $k = \frac{\pi}{2}$

(ii) $g^{-1}(x) = \arcsin \ln(1+x)$

(iii) $[e^{-1} - 1, e - 1]$

14 a) Proof

b) $\{2, 4, 8, 10, 14\}$ and $\{6, 12\}$

15 a) Student explanation

b) $g^{-1}(u, v) = (-u + 2v, 2u - 3v)$

c) (i–ii) Neither

16 a) Proof

b) $3n - 2; 3n - 1; 3n; n \in \mathbb{Z}^+$

17 Proof

18 The equivalence class of (1, 1) is a pair of straight lines through the origin with slopes ±1.

19 a) Not an equivalence relation

b) (i) Proof

(ii) $x + y = 2, x \neq y$

(iii) 1

20 a) Range is $\left[-\frac{9}{4}, \infty\right[$; not an injection

b) $g^{-1}(x) = +\sqrt{x + \frac{9}{4}} - \frac{1}{2}$ on $[0, 4]$

21 a) $]-1, 1[$

b) Proof

c) $f^{-1} = \ln\left(\frac{1+x}{1-x}\right)$

22 a) Proof

b) The equivalence classes are points lying, in the first quadrant, on straight lines through the origin.

23 a) R_4 is an equivalence relation.

b) The equivalence classes are $\{D, F\}$ and $\{E, G\}$.

Chapter 3

Exercise 3.1

1 a) Proof

b)

∘	0	2	4
0	0	2	4
2	2	4	0
4	4	0	2

c) Yes

2 a) (i) 75 (ii) 45 (iii) 8

(iv) 0 (v) 9 (vi) 3

(vii) 4608 (viii) 288

b) No; $x = 0$, $y = 0$, or $x = y$

c) No

3 Proof

4 e is the identity, s is the reflection with respect to the smaller diagonal, and l with respect to the larger diagonal, and r is a rotation of 180°.

∘	e	r	s	l
e	e	r	s	l
r	r	e	l	s
s	s	l	e	r
l	l	s	r	e

5 a)

$\circ$	p	r	s	t
p	p	p	p	p
r	p	r	s	t
s	t	s	r	p
t	t	t	t	t

b) r is the identity.

c) No d) r, s e) No

6 a)

$\circ$	p	r	s	t
p	p	r	s	t
r	r	p	t	s
s	s	s	s	s
t	t	t	t	t

b) p is the identity.

c) No d) p, r e) No

7 A group with identity 1 and each element is self-inverse.

8 Not a group: $1 + 1 = 2 \notin \{-1, 0, 1\}$.

9 A group with identity 0 and inverse defined by $(10k)^{-1} = -10k$.

10 A group with identity 1 and inverse defined by $\left(2^m\right)^{-1} = 2^{-m}$.

11 A group with identity 1 and inverse defined by $\left(2^m 3^n\right)^{-1} = 2^{-m}3^{-n}$.

12 A group with identity $f(x) = 0$ and inverse defined by $f^{-1}(x) = -f(x)$.

13 A group with identity 0 and inverse defined by $a^{-1} = -\dfrac{a}{a+1}$.

14 A group with identity 1 and inverse defined by $\left(a + b\sqrt{2}\right)^{-1} = \dfrac{a}{a^2 - 2b^2} - \dfrac{b}{a^2 - 2b^2}\sqrt{2}$.

15 Proof

16 Proof

17 a) 24

b) If we let $1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$, $a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$,

$b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$, $c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$, ..., then the table will look like this:

$\circ$	1	a	b	c	$\cdots$
1	1	a	b	c	$\cdots$
a	a	1	c	b	$\cdots$
b	b	d	1	f	$\cdots$
c	c	f	a	d	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

c) For example: $a \circ b = c \neq b \circ a = d$

18 a–c) Proof

19 a–b) Proof c) Yes; 3, 11

20

$\circ$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

21

$\propto$	w	x	y	z
w	y	z	w	x
x	z	w	x	y
y	w	x	y	z
z	x	y	z	w

22 Proof

23 a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$

e) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$ f) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$

g) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$ h) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$

24 a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$

e) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$ f) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$

g) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$ h) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$

25 Proof

26 Proof

27 a–b) Proof

28 Proof

29 Proof

30 Proof

31 29

Practice questions

1 Proof

2 a–b) Proof

3 a–b) Proof

4 a–b) Proof

5 a (i) Proof

(ii) $a = 3,\ b = -\dfrac{3}{2}$

b (i) $A = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(ii) $\left\{A, A^2, A^3, I\right\}$

6 a–b) Proof

7 a)

$*$	a	b	c	d
a	b	c	d	a
b	c	d	a	b
c	d	a	b	c
d	a	b	c	d

b) (i) $x = d$

(ii) $x = a$

8 a) Proof

b) R is an equivalence relation.

9 a) The operation is commutative.

b) Proof

10 a) 6

b) (i) $p_2 p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 1 & 5 \end{pmatrix}$

(ii) They do not commute.

c) $\left(p_1^2 p_2\right)^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix}$

11 a–c) Proof

12 a) (i) Proof
(ii) $\{1, 5, 9, \ldots\}, \{2, 6, 10, \ldots\}, \{3, 7, 11, \ldots\}, \{4, 8, 12, \ldots\}$
b) (i–ii) Proof

13 a) (i) Not closed (ii) Commutative
(iii) Not associative
b) (i) $e = 2$ (ii) $\{1, 2, 3\}$

14 a) (i) Proof
(ii) $\{2, 8\}, \{1, 4, 9\}$
b) Proof

Chapter 4

Exercise 4.1

1 Proof

2 a–b) Proof c) $\{1, 13\}, \{1, 9, 11\}$

3 a) $\{x, x^2, x^3, x^4\}$
b) $\{x, x^5\}$
c) 7 has 6 generators, 10 has 3, 15 has 8, and 20 has 8. The number of generators is the number of numbers less than or equal to the group order and is relatively prime to it.

4 a) $\{I, R, R^2\}, \{I, L\}$ b) No

5 a) 12, ([1], 12), ([2], 6), ([3], 4), ([4], 3), ([5], 12), ([6], 2), ([7], 12), ([8], 3), ([9], 3), ([10], 6), ([11], 12). Factors of 12.
b) 4, ([3], 4), ([7], 4), ([9], 2). Factors of 4.
c) 4, ([5], 2), ([7], 2), ([11], 2). Factors of 4.
d) 8, ([3], 4), ([7], 4), ([9], 2), ([11], 2), ([13], 4), ([17], 4), ([19], 2). Factors of 8.
e) 8, $(r, 4), (r^2, 2), (r^3, 4), (L_1, 2), (L_2, 2), (L_3, 2), (L_4, 2)$. Factors of 8.

6 a) $(U(3), 2), (U(4), 2), (U(12), 4)$
b) $(U(5), 4), (U(7), 6), (U(35), 24)$
c) $(U(4), 2), (U(5), 4), (U(20), 8)$
d) $(U(3), 2), (U(5), 4), (U(15), 8)$
$|U(mn)| = |U(m)| \cdot |U(n)|$; $(U(4), 2), (U(10), 4), (U(40), 16)$; $|U(mn)| = |U(m)| \cdot |U(n)|$ iff m and n are relatively prime.

7 3 or 6

8 $|a^2| = 3, |a^3| = 2, |a^4| = 3, |a^5| = 6$.
$|b^2| = 9, |b^3| = 3, |b^4| = 9, |b^5| = 9, |b^6| = 3, |b^7| = 9, |b^8| = 9$.

9 a) 2 and 6 generate $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$; 3 and 4 generate $\{1, 3, 4, 5, 9\}$; 10 generates $\{1, 10\}$.
b) Yes

10 a–b) Proof

11 a–b) Proof

12 Proof

13 Proof

14 a–b) Proof
c) Yes; 2 or 4; $\{1, 7\}, \{1, 9\}, \{1, 11\}, \{1, 15\}, \{1, 3, 9, 11\}, \{1, 5, 9, 13\}$
d) No

15 a) n
b) Proof

16 Proof

17 a) $\{1, x, x^2, y, xy, x^2y\}$
b) $\{1, y\}, \{1, xy\}, \{1, x^2y\}, \{1, x, x^2\}$

18 a) $1, x, x^2, y, xy, yx^2, yx, x^2y, xyx, yxy, x^2yx, xyx^2$
b) $\{1\}, \{1, y\}, \{1, x^2yx\}, \{1, xyx^2\}, \{1, x, x^2\}, \{1, xy, yx^2\}, \{1, yx, x^2y\}, \{1, xyx, yxy\}$

19 Proof

20 Proof

21 Proof

22 No. Only if $H \subseteq K$ or $K \subseteq H$.

23 Proof

24 $\{1, 2, 4\}, \{1, 6\}; \{1, 3\}, \{1, 5\}, \{1, 7\}; \{1, 4\}, \{1, 11\}, \{1, 14\}, \{1, 2, 4, 8\}, \{1, 4, 7, 13\}$

25 Proof

26 $\left\{\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, k \in \mathbb{N}\right\}, \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$

27 Proof

28 Proof

29 Proof

30 Proof

31 Proof

32 Proof

33 Generators: 8, 12

34 Not cyclic

35 Proof

Practice questions

1 a) 6
b) $p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$
c) $\{p_0, p_1, p_2\}$

2 a–b) Proof c) No

3 a–c) Proof

4 a–c) Proof d) $\{1, 13\}, \{1, 9, 11\}$

5 a–c) Proof

6 a–b) Proof

7 a–b) Proof

8 a) Lagrange's theorem (see page 95)
b) Proof

9 a–b) Proof

10 a) Proof
b)

×	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

c) Order of 1 is 1; order of 3 is 4; order of 7 is 4; order of 9 is 2.
d) $1 \leftrightarrow 1, 3 \leftrightarrow i, 7 \leftrightarrow i$, and $9 \leftrightarrow -1$ (or $3 \leftrightarrow -i, 7 \leftrightarrow i$)

∘	U	H	V	K
U	U	H	V	K
H	H	U	K	V
V	V	K	U	H
K	K	V	H	U

b) Proof

c)

◊	1	−1	i	$-i$
1	1	−1	i	$-i$
−1	−1	1	$-i$	i
i	i	$-i$	−1	1
$-i$	$-i$	i	1	−1

d) Not isomorphic

12 a–c) Proof

13 a–d) Proof

14 a)

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

b)

*	a	b	c	d
a	b	a	d	c
b	a	b	c	d
c	d	c	a	b
d	c	d	b	a

15 a) $\begin{pmatrix} a & b & c & d \\ b & d & a & c \end{pmatrix}$

b) For example: $\begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}$; $\begin{pmatrix} a & b & c & d \\ b & a & c & d \end{pmatrix}$

c) $\begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}$

$\begin{pmatrix} a & b & c & d \\ b & c & d & a \end{pmatrix}$; $\begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix}$; $\begin{pmatrix} a & b & c & d \\ d & a & b & c \end{pmatrix}$

16 a) Proof

b) $f^{-1}(z) = \log_3(z)$

17 a) Proof

b) Not Abelian

c–e) Proof

18 a)

∘	f	g	h	j
f	f	g	h	j
g	g	f	j	h
h	h	j	f	g
j	j	h	g	f

b) $+_4$ is isomorphic with x_5. Corresponding elements are: $0 \leftrightarrow 1, 1 \leftrightarrow 2, 2 \leftrightarrow 4, 3 \leftrightarrow 3$; or $0 \leftrightarrow 1, 1 \leftrightarrow 3, 2 \leftrightarrow 4, 3 \leftrightarrow 2$.

19 a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

b) (i) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has order 2, $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ has order 3,

$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ has order 3

(ii) $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right\}$

20 a)

*	1	3	4	9	10	12
1	1	3	4	9	10	12
3	3	9	12	1	4	10
4	4	12	3	10	1	9
9	9	1	10	3	12	4
10	10	4	1	12	9	3
12	12	10	9	4	3	1

b) Proof

c) 1 is of order 1; 12 is of order 2; 3 and 9 are of order 3; 4 and 10 are of order 6.

d) {1}, {1, 12}, {1, 3, 9}, {1, 3, 4, 9, 10, 12}

21 a–b) Proof   c) $\{p^2, pq\}$

22 a) Proof   b) 2

c (i) Proof

(ii) 0, with order 2

23 a) Student explanation b) $x = 8$

c) (i) Proof

(ii) {1, 8}, {1, 4, 7}

24 a) $\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -a & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) Proof   c) Proof

25 a) (i)

$\bullet_8$	**1**	**3**	**5**	**7**
1	1	3	5	7
3	3	1	**7**	5
5	**5**	7	1	3
7	7	5	3	**1**

(ii)

$*_{15}$	3	6	9	12
3	9	3	12	6
6	3	6	9	12
9	12	9	6	3
12	6	12	3	9

b) Not isomorphic

26 a) (i) c is the identity of ∘, b is the identity of ×

(ii) For ∘, a, b, d have order 2, c has order 1.
For ×, a, d have order 4, c has order 2, b has order 1.

b) (i) $\{a, c\}$; $\{b, c\}$; $\{c, d\}$

(ii) $\{b, c\}$

c) $x = b$ or $x = c$

27 a–b) Proof

28 a)

*	P	Q	R	T
P	P	Q	R	T
Q	Q	R	T	P
R	R	T	P	Q
T	T	P	Q	R

b) (i) Proof

(ii) $e = 0$. There is no inverse for $-\frac{1}{s} = t$.

(iii) It is an Abelian group. $\left\{0, -\frac{2}{s}\right\}$.

29 a) cis 0 (1), cis $\frac{\pi}{3}$, cis $\frac{2\pi}{3}$, cis $\pi\,(-1)$, cis $\frac{4\pi}{3}$, cis $\frac{5\pi}{3}$

b) (i–ii) Proof

(iii) The group of the integers 0, 1, 2, 3, 4, 5 under addition modulo 6; $m \to \text{cis}\,\dfrac{m\pi}{3}$

30 a)

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

b) Proof

c)

Number	0	1	2	3	4	5
Order	1	6	3	2	3	6

d) Generators: 1 and 5

e) {0, 2, 4}

f) {0},{0, 3}

31 a)

$*$	1	2	3	4	5	6
1	1	2	3	3	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

b) (i)

Number	1	2	3	4	5	6
Order	1	3	6	3	6	2

(ii) {1, 6}; {1, 2, 4}

c) $x = 2$ or 5

32 Proof

Option 3

Chapter 1

Exercise 1.1

1 Converges to 0
2 Converges to 2
3 Converges to 0
4 Diverges
5 Converges to 0
6 Converges to 0
7 Diverges
8 Diverges
9 Converges to $\sqrt{2}$
10 Converges to 1
11 Diverges
12 Converges to 1
13 Converges to 0
14 Converges to 1
15 Converges to 1
16–18 Proof
19 $\frac{1}{2}$
20 2
21 $\frac{1}{2}$
22 Converges to π
23 -1
24 $-\frac{1}{3}$
25 $\frac{1}{6}$
26 $\frac{1}{3}$
27 $\ln 2$
28 $\ln\left(\frac{a}{b}\right)$
29 1
30 Divergent
31 $\frac{1}{2}$
32 π
33 $\frac{1}{2}$
34 Divergent
35 $\ln 2$
36 2
37 k
38 a) Area increases without bound, i.e. infinite
b) π units^3
c) The area of the region is infinite; however, the volume of the solid created by rotating the region about the x-axis is finite.

Chapter 2

Exercise 2.1

1 a) 8 b) -1 c) 25
2 a) $\frac{3}{4}$ b) $\frac{3}{4}$ c) $\frac{1}{1+x}$
3 $\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{5}}+\frac{3}{\sqrt{10}}+\frac{4}{\sqrt{17}}+\dots$; diverges by nth term divergence test
4 $3+\frac{3}{4}+\frac{3}{16}+\frac{3}{64}+\dots$; converges to 4
5 $0+\ln\frac{1}{2}+\ln\frac{1}{3}+\ln\frac{1}{4}+\dots$; diverges by nth term divergence test
6 $\frac{3}{2}-\frac{3}{4}+\frac{3}{8}-\frac{3}{16}+\dots$; converges to 1
7 $\frac{1}{3}+\frac{2}{9}+\frac{2}{9}+\frac{8}{27}+\dots$; diverges by nth term divergence test
8 $-1+1-1+1-\dots$; diverges by nth term divergence test
9 $\frac{5}{11}+\frac{7}{16}+\frac{3}{7}+\frac{11}{26}+\dots$; diverges by nth term divergence test
10 $\frac{1}{e}+\frac{1}{e^2}+\frac{1}{e^3}+\frac{1}{e^4}+\dots$; converges to $\frac{1}{e-1}$
11 a) $\int xe^{-x}\,dx = -e^{-x}(x+1)+C$
b) $\int_1^\infty xe^{-x}\,dx = \frac{2}{e}$ and therefore the series is convergent.
12 a) Divergent b) Convergent
13–14 Proof
15 For $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, $\lim_{n\to\infty} a_n = 0$ but it is a p-series with $p=\frac{1}{2}\leqslant 1$ so the series diverges.
16 Proof
17 Converges
18 Diverges
19 Converges
20 Converges
21 Converges
22 Diverges
23 Diverges
24 Diverges
25 Diverges
26 Diverges
27 Converges
28 Diverges
29 Converges
30 Converges
31 Diverges
32 5
33 a) $S_4 = \frac{10\,016}{11\,025}\approx 0.908\,48$; error $< \frac{1}{81}$
b) $S_4 = 0.095\,308\bar{3}$; error $< 0.000\,006$
34 a) $(n+1)^2+1$
b) $\int_1^\infty \frac{1}{(x+1)^2+1}\,dx = \lim_{b\to\infty}[\arctan(x+1)]_1^b = \frac{\pi}{2}-\arctan(2)$
$=\arctan\left(\frac{1}{2}\right)$; since $\int_1^\infty \frac{1}{(x+1)^2+1}\,dx$ converges to $\arctan\left(\frac{1}{2}\right)$, then $\sum_{n=1}^{\infty}\frac{1}{n^2+2n+2}$ must also converge.
35 Diverges
36 a) 1.202 606 481 with error $< 0.006\bar{1}$ b) 10 terms
37 11 terms
38 $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$ is conditionally convergent.
39 Converges absolutely
40 Converges conditionally
41 Diverges
42 Converges conditionally

43 Converges absolutely **44** Converges absolutely

45 $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}-\frac{1}{4}+\frac{1}{7}+\frac{1}{9}-\frac{1}{8}+\ldots$; the sum of this series is 1. The terms of the alternating harmonic series are rearranged such that consecutive positive terms are added until the sum is greater than 1, then consecutive negative terms are added until the sum is less than 1, and so on. Note that the difference between the partial sums and 1 is less than the last term used, so the series converges to 1.

46 7 terms **47** Proof

Chapter 3

Exercise 3.1

1 $R=1;\ -1\leqslant x<1$ **2** $R=1;\ 1<x<3$

3 $R=2;\ 2\leqslant x<4$ **4** $R=\infty;\ x\in\mathbb{R}$

5 $R=1;\ -1\leqslant x\leqslant 1$ **6** $R=1;\ 1\leqslant x\leqslant 3$

7 $R=1;\ 0<x<2$ **8** $R=1;\ -1\leqslant x<1$

9 $R=0;\ x=0$ **10** $R=\frac{4}{3};\ -\frac{4}{3}\leqslant x<\frac{4}{3}$

11 $R=4;\ -4<x<4$ **12** $R=3;\ -3\leqslant x\leqslant 3$

13 $R=e;\ -e<x<e$ **14** $R=0;\ x=4$

15 $-\frac{1}{k}<x<\frac{1}{k}$

16 a) $\sum_{n=0}^{\infty}(-1)^n x^n;\ -1<x<1$

b) $A=\frac{1}{2},\ B=\frac{1}{2}$

c) $\sum_{n=0}^{\infty}x^{2n};\ -1<x<1$

17 a) $e^{-x^2}=1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\ldots+(-1)^n\frac{x^{2n}}{n!}+\ldots;\ R=\infty$

b) $\int e^{-x^2}dx=\int\left(1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)\,n!}+\ldots\right)$

$=x-\frac{x^3}{3\cdot 1!}+\frac{x^5}{5\cdot 2!}-\frac{x^7}{7\cdot 3!}+\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)n!}+\ldots;$

radius of convergence is also $R=\infty$.

c) $\int_0^1 e^{-x^2}\,dx\approx 1-\frac{1}{3}+\frac{1}{10}-\frac{1}{42}+\frac{1}{216}=\frac{5651}{7560}\approx 0.747;$

error $<a_6=\frac{1}{11\cdot 5!}=0.000\overline{75}<0.001$

18 a) $x^2-\frac{x^4}{3!}+\frac{x^6}{5!}$ b) $x+\frac{1}{3}x^3+\frac{2}{15}x^5$

c) $x-\frac{1}{2}x^2+\frac{7}{6}x^3$

19 $\sum_{n=0}^{\infty}nx^{n-1}$ for $-1<x<1$

20 a) $\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+2}}{n!}$ b) Proof

21 a) $\sin x\approx x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}$

b) $\sin\left(\frac{\pi}{12}\right)\approx 0.258\,819$

c) Error $<1.4165\times 10^{-10}$

22 $-\frac{1}{2}<x<\frac{1}{2}$

23 $(x-1)\,e+(x-1)^2\,e+\frac{(x-1)^3}{2}\,e+\frac{(x-1)^4}{6}\,e$

24 $\sum_{n=1}^{\infty}\frac{2}{(2n-1)}x^{2n-1}=2x+\frac{2x^3}{3}+\frac{2x^5}{5}+\ldots$

25 a) $\sum_{n=0}^{\infty}(-1)^n\,x^{2n}=1-x^2+x^4-x^6+\ldots$

b) Proof c) Proof d) $\pi\approx 2.976$; error $<0.142\,86$

26 a) $f^{(n)}(x)=\frac{e^x+(-1)^n\,e^{-x}}{2}$

b) $f(x)=1+\frac{x^2}{2}+\frac{x^4}{24}+\ldots$

c) $f\left(\frac{1}{2}\right)\approx\frac{433}{384}=1.127\,604\,1\overline{6}$

d) Error $<0.000\,136$

27 $-1.59<x<1.59$

28 $xe^x=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}=x+x^2+\frac{x^3}{2!}+\frac{x^4}{3!}+\ldots$

29 $\sec^2 x=1+x^2+\frac{2x^4}{3}+\frac{17x^6}{45}+\frac{62x^8}{315}+\ldots$

30 a) $\sum_{n=0}^{\infty}\frac{e^2}{n!}(x-2)^n$

b) $\sum_{n=0}^{\infty}(-1)^n\frac{(x^3)^{2n+1}}{(2n+1)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+3}}{(2n+1)!}$

c) $-\frac{1}{2}\sum_{n=0}^{\infty}(n+1)\,nx^{n-1}$

d) $\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(x-1)^{n+4}$

31 a) 1 b) $\frac{1}{3}$

Practice questions

1 $\ln(\cos x)\approx-\frac{x^2}{2}-\frac{x^4}{12}$

2 a) $\sin^2 x\approx x^2-\frac{x^4}{3}$ b) $\cos^2 x\approx 1-x^2+\frac{x^4}{3}$

3 $e^x\sin x\approx x+x^2+\frac{x^3}{3}$

4 $e^{3x}\approx 1+3x+\frac{9x^2}{2}+\frac{9x^3}{2}$

5 $\sec x\approx 1+\frac{x^2}{2}+\frac{5x^4}{24}$

6 a) $e^x\approx 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}$

b) $e^x\approx 1+x^2+\frac{x^4}{2!}$

c) $e^x\approx 1+x+\frac{3x^2}{2}+\frac{7x^3}{6}+\frac{25x^4}{24}$

7 $\ln(2+3x)=\ln 2+\frac{3}{2}x-\left(\frac{3}{2}\right)^2\frac{x^2}{2}+\left(\frac{3}{2}\right)^3\frac{x^3}{3}-\left(\frac{3}{2}\right)^4\frac{x^4}{4}+\ldots;$

$R_n(x)=\frac{(-1)^n\,3^{n+1}}{(n+1)\,(2+3c)^{n+1}}x^{n+1}$

8 a) $\sqrt{4+x}\approx 2+\frac{x}{4}-\frac{x^2}{64}+\frac{x^3}{512}-\frac{5x^4}{16\,384}$

b) $R_4(x) = \dfrac{7}{256\,(4+x)^{9/2}}\,x^5$; since $2^9 < (4+0.1)^{9/2}$ then

$0 \leqslant R_4(x) \leqslant \dfrac{7}{256 \cdot 2^9}(0.1)^5 < 5.34 \times 10^{-10}$

9 2 terms needed; 0.996 195

10 a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots$

b) $\displaystyle\int_0^1 e^{-x^2}\,dx \approx \frac{23}{30}$

c) Error $< \dfrac{e}{42}$

11 a) $\displaystyle\frac{1}{1+x^2} = \sum_{n=1}^{\infty} (-1)^{n+1} x^{2n-2}$

b) Proof c) Proof d) $\dfrac{\pi}{4}$

12 a) $\dfrac{1}{1+x} \approx 1 - x + x^2 - x^3 + \ldots$ and

$\dfrac{1}{1-x} \approx 1 + x + x^2 + x^3 + \ldots$

b) $\dfrac{-3}{x-2} + \dfrac{4}{x-3}$

c) $\dfrac{x+1}{x^2-5x+6} \approx \dfrac{1}{6} + \dfrac{11x}{36} + \dfrac{49x^2}{216} + \dfrac{179x^3}{1296} + \ldots$

13 a) $\dfrac{1}{1-x}$

b) $\displaystyle\sum_{n=1}^{\infty}\left[-(x+1)^{n-1}\right] = -1 - (x+1) - (x+1)^2 - (x+1)^3 - \ldots,$

$-2 < x < 0$

14 a) Series converges by the ratio test
b) Series converges by the integral test
c) Series converges by the alternating series test

15 $\dfrac{4}{x} - \dfrac{1}{x-1}$

16 a) Series converges by the ratio test
b) Series diverges by the integral test

17 a) Proof b) $k = -\dfrac{1}{32}$
c) Proof d) 3.1550

18 a) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$ b) 0.3103

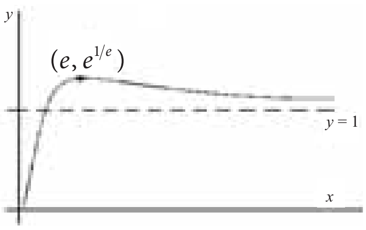
19 a) $R = 1$ b) $4 \leqslant x \leqslant 6$

20 Converges by the alternating series test; sum ≈ 0.63

21 a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ b) Proof

22 a) Converges by comparison test
b) Converges by alternating series test

23 a) Proof
b)



c) $P_2(x) = e^{1/e} - \dfrac{e^{1/e-3}}{2}(x-e)^2$, which is a parabola with vertex $(e, e^{1/e})$.

24 Diverges by comparison with the harmonic series

25 a) $\displaystyle S_{2n} = \sum_{k=1}^{2n} \frac{1}{k} = S_n + \sum_{k=n+1}^{2n} \frac{1}{k} \geqslant S_n + \sum_{k=n+1}^{2n} \frac{1}{2k} = S_n + \frac{1}{2}$

b) Proof

26 a) $\displaystyle S_{2n} = \sum_{k=1}^{2n} u_k = \sum_{k=1}^{n}(u_{2k-1} + u_{2k}) = \sum_{k=1}^{n}\left(\frac{3}{2k+1} - \frac{1}{2k}\right)$

$\displaystyle = \sum_{k=1}^{n} \frac{4k-1}{2k\,(2k+1)}$

b) Divergent by limit comparison test

27 a) Integral test for $\sum a_n$: Let $a_n = f(n)$, where $f(x)$ is a continuous, positive and decreasing function for all $x \geqslant N$ and N is some positive integer. Then the series $\displaystyle\sum_{n=N}^{\infty} a_n$ and the integral $\displaystyle\int_N^{\infty} f(x)\,dx$ both diverge or both converge. That is, if the integral is finite then $\sum a_n$ is finite, and if the integral is infinite then $\sum a_n$ is infinite.

b) Converges by integral test

28 a) (i) $\sin x \approx x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!}$

(ii) $e^{x^2} \approx 1 + x^2 + \dfrac{x^4}{2!} + \dfrac{x^6}{3!}$

b) $e^{x^2} \sin x \approx x + \dfrac{5}{6}x^3 + \dfrac{41}{120}x^5$

c) $\dfrac{5}{6}$

29 $\ln(1 + \sin x) \approx x - \dfrac{x^2}{2!} + \dfrac{x^3}{3!} - \dfrac{2x^4}{4!}$

30 Ratio test gives interval of convergence as $-1 \leqslant x < 1$.

31 Proof

32 $\displaystyle\lim_{x\to 0}\left(\frac{\sin x - x}{x \sin x}\right) = \lim_{x\to 0}\left(\frac{\cos x - 1}{\sin x + x\cos x}\right) =$

$\displaystyle\lim_{x\to 0}\left(\frac{-\sin x}{2\cos x - x\sin x}\right) = 0$

33 Proof

34 a) (i) $y' = \dfrac{\cos x}{1+\sin x}$; $y'' = -\dfrac{1}{1+\sin x}$;

$y^{(3)} = -\dfrac{\cos x}{(1+\sin x)^2}$;

$y^{(4)} = \dfrac{-\sin x\,(1+\sin x)^2 - 2\,(1+\sin x)\cos^2 x}{(1+\sin x)^4}$

(ii) Proof

b) (i) $\ln(1-\sin x) = \ln(1+\sin(-x)) = x - \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3 - \dfrac{1}{12}x^4 + \ldots$

(ii) $\ln(\cos x) = -\dfrac{1}{2}x^2 - \dfrac{1}{12}x^4 + \ldots$

(iii) $\tan x = x + \dfrac{1}{3}x^3 + \ldots$

c) -2

35 $R = \dfrac{1}{4}$

36 a) (i) Converges by alternating series test

(ii) $S_4 = 1 - \dfrac{1}{3!} + \dfrac{1}{5!} - \dfrac{1}{7!} \approx 0.841\,468$

(iii) Error $< 0.000\,002\,76 = 2.76 \times 10^{-6}$

b) (i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

(ii) $a_n = (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$

(iii) Proof

(iv) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

37 a) $\frac{1}{2}$ b) 0

38 a) (i) Proof

(ii) Converse is not true; a counter example is $\sum \frac{1}{n^2}$ which is convergent but $\sum \frac{1}{n}$ is not.

b) (i) $k > 1$ (ii) $k \leqslant 1$

39 a) (i) Proof

(ii) $a_n = \frac{1^2 \times 3^2 \times \ldots (n-2)^2}{n!}$, for odd $n \geqslant 3$

b) $R = 1$

c) $\pi \approx 3.139$

40 a) $\lim_{n\to\infty} a_n = \frac{1}{2}$ b) $N = 1501$

41 $-1.59 < x < 1.59$

42 Convergent by comparison with geometric series $\sum 2^{-n}$

43 a) Proof b) Proof

c) $S = e - 1$

44 a) Proof b) Proof

c) GDC value: $\ln(1.5) \approx 0.405\,465\,1081$

For $x = 1.5$: $\frac{x^3}{3} - \frac{3x^2}{2} + 3x - \frac{11}{6} = 0.41\bar{6}$.

For $x = 1.5$: $\frac{x^3}{12} - \frac{x^2}{2} + \frac{3x}{2} - \frac{5}{6} - \frac{1}{4x} = 0.406\,25$.

The second approximation is nearer to the true value.

45 a) (i) $I_n = \frac{1}{2}\ln\left(\frac{1+\alpha^2 n^2}{1+n^2}\right)$

(ii) $\lim_{n\to\infty} I_n = \frac{1}{2}\ln(\alpha^2)$ or $\ln\alpha$

b) -2

46 a) (i) 2 (ii) 2

b) The argument is incorrect because the denominator is not zero when $x = 3$.

47 $a = 1,\ b = 1,\ c = \frac{1}{2},\ d = 0$

48 a) Convergent by comparison test, or limit comparison test

b) (i) $\frac{1}{2n} - \frac{1}{2(n+2)}$ (ii) $\frac{3}{4}$

49 $-1 \leqslant x < 1$

50 a) $-\frac{5}{8}$ b) $-\frac{1}{2}$

51 a) $A = \frac{1}{4},\ B = -\frac{1}{4}$

b) $S_n = \frac{1}{4}\left(\frac{1}{3} - \frac{1}{4n+3}\right)$

c) $\sum_{r=1}^{\infty}\left(\frac{1}{16r^2+8r-3}\right) = \lim_{n\to\infty}\frac{1}{4}\left(\frac{1}{3} - \frac{1}{4n+3}\right) = \frac{1}{12}$; hence, the series is convergent.

52 $p > 1$

53 a) $\ln 2$ b) -3

54 a) (i) $f^{(n)}(x) = \frac{e^x + (-1)^n e^{-x}}{2}$

(ii) $f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4} + \ldots$

(iii) $f\left(\frac{1}{2}\right) \approx 1 + \frac{1}{8} + \frac{1}{384} = \frac{433}{384}$

(iv) Error $< 0.000\,136$

b) Series is convergent by integral test

55 a) (i) Domain $[-1, 1]$, range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\arcsin x = x + \frac{x^3}{6} + \ldots$

b) $\cos(\arcsin x) = 1 - \frac{x^2}{2} - \frac{x^4}{8}$

c) (i) $p^r\left(1 + \frac{q}{p}x^2\right)^r = p^r\left(1 + r\frac{q}{p}x^2 + \frac{r(r-1)}{2}\frac{q^2}{p^2}x^4\right)$

(ii) $p = 1, q = -1,\ r = \frac{1}{2}$; hence, the series in b) and c) is $(1-x^2)^{1/2}$ since $\cos(\arcsin x) = \cos\left(\arccos\sqrt{1-x^2}\right) = (1-x^2)^{1/2}$.

56 a) 2 b) -1 c) Proof

57 a) (i) Proof (ii) Proof (iii) $p = \frac{2}{3},\ q = \frac{4}{3}$

b) Proof c) Proof

58 Series is convergent by the ratio test

59 a) Proof b) $1.18 < \sum_{n=1}^{\infty}\frac{1}{n^3} < 1.22$

60 a) 0

b) $f'(x) = \frac{1}{1-x},\ f''(x) = \frac{1}{(1-x)^2},\ f'''(x) = \frac{2}{(1-x)^3}$

c) $\ln 2 \approx \frac{1}{2} + \frac{1}{8} + \frac{1}{24} = \frac{2}{3}$

d) Error < 0.25

e) Actual error $= \ln 2 - \frac{2}{3} \approx 0.026\,48$; upper bound for error found in d) is much larger than the actual error, so the estimate of $\frac{2}{3}$ found in c) cannot be considered a good estimate.

61 a) $\frac{1}{2\pi}$ b) -2

62 a) Proof

b) Using $x = \frac{\pi}{3}$: $\ln 2 \approx \frac{\pi^2}{18} + \frac{\pi^4}{972}$; using $x = \frac{\pi}{4}$: $\ln 2 \approx \frac{\pi^2}{16} + \frac{\pi^4}{1536}$.

63 a) $R = 3$

b) Converges by limit comparison test (comparing to convergent series $\sum_{n=1}^{\infty}\frac{1}{n^2}$)

64 a) Proof

b) $\ln(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \ldots$

c) $\ln(1 - \sin x) = -x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} - \ldots$

d) Proof

e) 0

65 $-\dfrac{8}{315}$

66 Given $\sum_{n=1}^{\infty} a_n$ and $a_n = f(n)$, integral test can be applied if $f(x)$ is positive, continuous and decreasing for $x > 1$; $\sum_{n=1}^{\infty} n\, e^{-n^2}$ converges.

67 $8 + 17(x-1) + 16(x-1)^2 + 7(x-1)^3 + (x-1)^4$

68 a) 0 for $-1 \leqslant \sin n \leqslant 1$

b) $\dfrac{\pi}{2}$ c) $\dfrac{2}{5}$

d) 0 e) 0 f) -1

69 a) Converges; geometric series with $r = \dfrac{1}{1.1}$, so $|r| < 1$.

b) Diverges by nth term divergence test

c) Converges by limit comparison test

d) Diverges by integral test

e) Converges; comparison test (compare to p-series with $p = 3$)

70 a) $\sin(\pi x) \approx 1 - \dfrac{\pi^2 (x - \frac{1}{2})^2}{2!} + \dfrac{\pi^4 (x - \frac{1}{2})^4}{4!} - \ldots$

b) 0.924

Chapter 4

Exercise 4.1

1 (i) c (ii) a (iii) d (iv) b

2 a) $2x^2 - y^2 = C$ b) $y = \dfrac{x}{1 - Cx}$

c) $\ln(y-1) - \ln y + C_1 = -\dfrac{1}{x}$ or $\dfrac{y}{y-1} = C_2 e^{1/x}$

d) $x = C_1 \sin y$ or $y = \arcsin(C_2 x)$

e) $y = Ce^{x^2/2}$ f) $y^2 = 2\sqrt{x^2+1} + C$

g) $\ln\sqrt{\dfrac{y-1}{y+1}} = e^x + C$ h) $x = y \ln y - y + C$

3 $\int \dfrac{y+1}{y}\,dy = \int \dfrac{x+1}{x}\,dx \Rightarrow \int\left(1 + \dfrac{1}{y}\right)dy = \int\left(1 + \dfrac{1}{x}\right)dx$

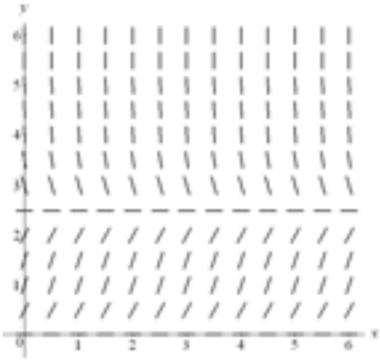
$\Rightarrow y + \ln|y| = x + \ln|x| + C$

$e^{y + \ln y} = e^{x + \ln x + C} \Rightarrow e^{\ln y} e^y = e^{\ln x} e^x e^C \Rightarrow ye^y = Axe^x$

4 $y = \pm\sqrt{2 \sin x + C}$

The constant C cannot be completely arbitrary because $2 \sin x + C \geqslant 0$. If $C < -1$, then $2 \sin x + C$ will always be negative, regardless of the value of x. If $C > 1$, then $2 \sin x + C$ will always be positive. If $-1 \leqslant C \leqslant 1$, then whether $2 \sin x + C$ is positive or negative will depend on the value of x.

5 a)



b) $\dfrac{5}{2}$ c) $\dfrac{5}{2}$

d) Regardless of the initial value of the population, as time increases, the population stabilizes at 2 500.

6 $y = -\sqrt{x^2 + \tan x + 25}$

7 a) Proof

b) $y = \dfrac{x+1}{x-1}$

8 (i) b (ii) d (iii) c (iv) a

9 $y = \dfrac{7x+1}{7-x}$

10 a) $\dfrac{1}{3(x-2)} - \dfrac{1}{3(x+1)}$

b) proof

11 a) $y = C(x^2 - 1) + 1$

b) $\dfrac{dy}{dx} + \left(\dfrac{2x}{1-x^2}\right)y = \dfrac{2x}{1-x^2}$; integrating factor is $\left|\dfrac{1}{1-x^2}\right|$; leads to same solution as in part a)

12 a) $y = x^4 + \dfrac{C}{x^2}$ b) $y = Ce^{x^2/2} - 1$

c) $y = \dfrac{1}{3}x^4 + Cx$ d) $y = xe^{\cos x} + Ce^{\cos x}$

e) $y = xe^{x^3} + Ce^{x^3}$ f) $y = x \ln|x| + Cx$

13 $y = x \csc x + C \csc x$

14 a)–c) Proof

d) $y = \dfrac{x}{2} + \dfrac{\arcsin x}{2\sqrt{1-x^2}} + \dfrac{1}{\sqrt{1-x^2}}$

15 a)–b) Proof

c) $y = \tan x + C \sec x$

16 $y = \dfrac{1}{3}x^2 + \dfrac{C}{x}$

17 $y = \dfrac{1}{3}x^2 \ln x - \dfrac{1}{9}x^2 + \dfrac{10}{9x}$

18 $C = \dfrac{y - x}{(y + x)^2}$

19 a) $y = Cx + C$ b) $y = Cx^2 - x$

c) $y = Cx^3 - x$ d) $2x^3 + 3xy^2 + 3y^3 = C$

e) $y^2 = \dfrac{x^2}{2} - \dfrac{C}{x^2}$ f) $y = x \ln(Cxy)$

20 a) Proof b) $x^2 + 4xy - 3y^2 - 1 = 0$

21 Proof

22 Proof

23 a) $\left|\dfrac{y}{y+1}\right| = C|x|$ b) $\left|\dfrac{y}{y+1}\right| = \dfrac{1}{2}|x|$

c)

x_n	y_n
1.2	1.400
1.4	1.960
1.6	2.789
1.8	4.110

d)

x_n	approx. y_n	exact y_n	% error
1.2	1.400	1.5	$6.\overline{6}$
1.4	1.960	$2.\overline{3}$	16
1.6	2.789	4	30.3
1.8	4.110	9	54.3

24 $y \approx 1.5405$

25 $y \approx 5.9584$

26 $y^2 = Cx^3 - x^2$

27

x_n	y_n
1.1	4.2
1.2	4.425 43
1.3	4.677 87
1.4	4.959 04
1.5	5.270 81

28 a) Proof

b) $y(1) \approx 0.327\ 68$

c) $y(1) \approx 0.348\ 678\ 4401$

d) Actual value to 10 s.f. is $y(1) \approx 0.367\ 879\ 4412$; using more steps (and a smaller step size) gives a better approximation.

Practice questions

1 a) $y = (x+c)x^3$ b) $y = (x+1)x^3$

2 $y = \sqrt{\dfrac{2x^5}{5} + \dfrac{6x^2}{5} + \dfrac{3}{5}}$

3 $y = Ce - \dfrac{1}{x}$

4 $y = \dfrac{C}{x} + \dfrac{\sin x}{x} - \cos x$

5 a) $y = \dfrac{C}{x} + \dfrac{x^3}{4}$ b) $y = \dfrac{16}{x} + \dfrac{x^3}{4}$

6 a) 6 b) 1 c) 2 d) 3 e) 4 f) 5

7 $y = 8\sin^2 x - 2$

8 a) $y = -2x + 12$ b) $y = \dfrac{8x}{x+1}$

9 $y = \tan x + \sec x$

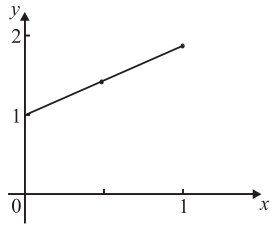
10 a) Proof b) $5x = \dfrac{y^2}{x^2} + 1$ $\left(\text{or } y = x\sqrt{5x-1}\right)$

11 Proof

12 a) $y \approx 1.84$

b) (i) $y = \sqrt{4 - x^2}$ (ii) $y \approx 1.77$

c)



Since $\dfrac{dy}{dx}$ is decreasing, the value of y is overestimated at each step.

13 a) $a = \dfrac{1}{2}$, $b = \dfrac{1}{2}$, $c = -\dfrac{1}{2}$

b) (i) $I = \dfrac{1}{2}\ln|1+x| + \dfrac{1}{4}\ln|1+x^2| - \dfrac{1}{2}\arctan x + k$

(ii) $C = \dfrac{3\pi}{8} - \dfrac{3}{4}\ln 2$

14 $y \approx 3.5$

15 a) Proof b) $\sec x$

c) $y = \sin x + 2\cos x$

16 a) Proof b) $y^2 = 6x^2 \ln x + Cx^2$

c) $y^2 = 6x^2 \ln x + 4x^2$

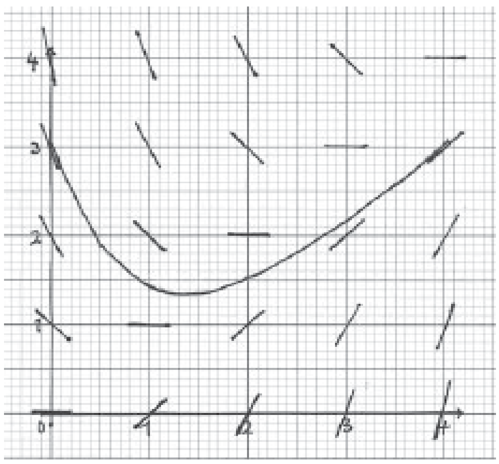
17 $y \approx 2.14$

18 a) Proof b) $y = \dfrac{3x^3 - x}{3x^2 + 1}$

19 a) Proof b) $e^{-\sin x}$

c) $y = -\sin x - 1 - e^{\sin x}$

20 a)–b)



c) $y = x - 1 + 4e$

21 a) (i) Proof

(ii) $y = 2 + \dfrac{2x}{1!} - \dfrac{2x^2}{2!} - \dfrac{10x^3}{3!} + \dfrac{2x^4}{4!} + \ldots$

b) $y(0.5) \approx 2.55$

c) $y(0.5) \approx 2.67$

d) Approximation using Maclaurin series can be made more accurate by computing more terms of the series; approximation using Euler's method can be made more accurate by decreasing the step value.

22 $yx^2 = \frac{1}{3}(1 + x^2)^{\frac{3}{2}} + \frac{1}{3}$

23 a) Proof

b) $\ln|x-1| = 3\arctan\left(\frac{y-2}{x-1}\right) - \frac{1}{2}\ln\left(1+\left(\frac{y-2}{x-1}\right)^2\right) + C$

24 a) $\frac{1}{x^2}$ b) $y = x^2\left(\arctan x + 1 - \frac{\pi}{4}\right)$

25 $\frac{2}{3}\ln 2$

26 a) (i) $y(1.3) \approx 2.14$ (ii) Decrease the step size

b) $y = x^2 + e^{1-x^2}$

27 $y = \left(\frac{x+2}{x+1}\right)\left(\ln(x+2) + \frac{1}{x+2} + C\right)$

28 $y = x^2(x^2+1) + C(x^2+1)$

29 $I = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$; $I = \frac{E}{R}e^{\frac{R}{L}t} - \frac{E}{R}$

30 $\sec(xy) = -2\ln(\cos x) + C$

$-x$

Option 4

Chapter 1

Exercise 1.1

1 9 **2** 30 **3–6** Proof

7 a) Q = 30, R = 8
b) Q = −6, R = 70
c) Q = −5, R = 25

8–15 Proof

16 a) Proof b) $q = -8, r = 1$

17–18 Proof

19 $x = 4, y = 8$ **20** $x = 3, y = 9$

21 $\varnothing$ **22** Proof

23 True **24** True

25 True **26** True

27 False **28** False

29 True

Exercise 1.2

1 4 **2** 1 **3** 17 **4** 68
5 77 **6** 1

7 $x = -17, y = 7$ **8** $x = -1, y = 1$

9 $x = -535, y = 132$ **10** $x = 9, y = 4$

11 $x = -1769, y = -29$ **12** $x = 5, y = 4$

13 No **14–16** Proof

17 8968 **18** 125 328

19 2100

20 (12, 360), (24, 180), (36, 120), (60, 72)

21 $\text{lcm}(a, b) = ab$ **22** No **23–30** Proof

Exercise 1.3

1–5 Proof

6 For example, they end with 1 or 7.
7 a) $3 \cdot 29$ b) 19^2 c) $3^3 \cdot 5 \cdot 7$
d) $7 \cdot 11 \cdot 13$ e) $2^4 \cdot 19 \cdot 23$

8 a) gcd = 1, lcm = $3 \cdot 29 \cdot 19^2$
b) gcd = 1, lcm = $19^2 \cdot 7 \cdot 11 \cdot 13$
c) gcd = 1, lcm = $3 \cdot 29 \cdot 19^2 \cdot 7 \cdot 11 \cdot 13$
d) gcd = 1, lcm = $2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 29$

9 6, 10, 15, 42, 70

10–12 Proof

13 $x \nmid y$, gcd = $3^2 \cdot 13$, lcm = $3^2 \cdot 5 \cdot 11^2 \cdot 13$

14 $x \nmid y$, gcd = $2^2 \cdot 23$, lcm = $2^3 \cdot 5^3 \cdot 23^2$

15 $x | y$, gcd = $3^2 \cdot 11 \cdot 23$, lcm = $3^2 \cdot 7 \cdot 11 \cdot 23$

16 $x \nmid y$, gcd = 5, lcm = $2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$

17–22 Proof

23 1, 3

24 1, 2

25 When a is odd, always; when a is even, only when c is even.

Chapter 2

Exercise 2.1

1 a) True b) False c) False d) True

2 19 **3** Proof **4** 2

5 5 **6** 17 **7** 9

8 38 **9** 19 **10** 5

11 6 **12** 6 **13** 15

14 12 **15** 1 **16** 16

17 5 **18** 11 **19–33** Proof

34 1, 18 **35** −18, 5, 28

36 3, 7, 11, 21, 33, 77, 231

37–39 Proof **40** 11, 39, 21 **41–42** Proof

Exercise 2.2

1 a) No solution b) Solution c) No solution

2 a) $x = 7 - 7t, y = 10 - 13t$
b) $x = 1 + 35t, y = -6 - 221t$
c) $x = -141 + 349t, y = 120 - 297t$

3 a) $x = 8 - 11t, y = 1 - 5t$, with $t \in \{..., -2, -1, 0\}$
b) No positive solutions
c) (1, 66), (12, 4)

4

Apples	16	34	52
Oranges	71	46	21

5 7 of the €4.98 posters and 11 of the €5.98 posters.

6 $10d + 25q = 455$; minimum = 20, maximum = 44

7

Chicken	3	10	17
Geese	9	5	1

8 (Calves, lambs, piglets): (5, 41, 54), or (10, 22, 68), or (15, 3, 82)

9 €3.96

10 23

11 Minimum number of sheep required = 16. Transaction is not possible.

12 $(1 + 2t, -1 - 3t)$ **13** $(1 - 2t, 1 - 3t)$

14 $(6 + 14t, -7 - 17t)$

15 $(1 - 4t, 2 - 11t)$ or $(1 + 4t, 2 + 11t)$

16 None **17** None

18 $(345 + 503t, -275 - 401t)$

19 $(6 + 7t, -11 - 13t)$
20 $(4 + 5t, -7 - 9t)$
21 $(5 + 11t, -3 - 7t)$
22 $(13 + 19t, -6 - 9t)$
23 $(1 + 3t, 16 - 2t), 0 \leqslant t < 8$
24 $(4 + 4t, 12 - 3t), 0 \leqslant t < 4$
25 $(3 + 3t, 8 - 2t), 0 \leqslant t < 4$
26 None
27 None
28 $(2 + 5t, 9999 - 3t)$
29 None
30 None
31 $(1 + 7t, 9 + 2t)$
32 $(3 + 17t, 2 - 22t)$
33 $(20 + 40t, -6 - 11t)$
34 (21, 19) or (72, 8)
35–36 Proof

Exercise 2.3

1 $2 + 7k$
2 $2 + 3k$
3 $33 + 40k$
4 $41 + 49k$
5 $111 + 888k$
6 $75 + 80k$
7 $5 + 7k$
8 $2 + 3k$
9 $16 + 24k$
10 No solution
11 $812 + 1001k$
12 $10 + 45k$
13 No solution
14 $k \in (0, 4, 8, 12, \ldots, 32\}; 4$
15 11 (mod 12)
16 151 (mod 414)
17 34 (mod 35)
18 13 (mod 55)
19 6 (mod 210)
20 559 (mod 1430)
21 (2 (mod 5), 2 (mod 5))
22 No solution
23 (k (mod 5), $2 + k$ (mod 5))
24 (k (mod 7), $4 + 4k$ (mod 7))

Exercise 2.4

1 $(5600)_7$
2 $(1071)_{10}$
3 $(1562773)_8$
4 $(235056)_{10}$
5 $(5018)_{10}$
6 $(11111011010)_2$
7 $(77F394FB)_{16}$
8 $(33047851104)_{10}$
9 $(479)_{16}$
10 $(74E)_{16}$
11 $(1111111011001101110)_2$
12 $(11111011110111110101100111011011000100 1)_2$
13 a) When n is even.
b) When either a is a multiple of 3 or n is a multiple of 3.
c) When a is even.

Exercise 2.5

1 9
2 3
3 5
4 10
5 10
6 3 (mod 17)
7 9 (mod 17)
8 9 (mod 17)
9 5 (mod 11)
10 9 (mod 13)
11 1
12 a) 8 (mod 11), 11 (mod 13), 10 (mod 17)
b) 1064 (mod 2431)
13 1
14 10
15 8
16–20 Proof

Practice questions

1 a) Proof
b) (i) Proof b) (ii) $x = 1, 4$, or 7
2 $x = -2, y = 3$
3 Proof
4 a) 8 b) $m = 11, n = 30$
5 a) Student's explanation
b) $d = 14; x = 3, y = -7$
6 a) $x = 11, y = -6$ b) Proof
7 Proof
8 a) $(23731)_8$ b) Proof c) Follows from b)
9 32
10 $x = 52 + 105k$
11 a) 16 b) $a = -12$ and $b = 5$
12 a) 235 b) 105 441 c) 9025
13 a) Student's explanation
b) $x = 1 - 2n, y = 1 - 3n$
14 a) 346 b) Proof
15 a) {1, 2, 3, 6} b) 6
c) $6k - 4$ or $6k - 2, k \in \mathbb{Z}^+$
16 a) 5
b) (i) Student's explanation (ii) (−18, 27)
(iii) $(-18 + 15m, 27 - 22m)$
17 Proof
18 a) 1
b) (i) $x = 119 - 73k, y = -70 + 43k$ (ii) (−27, 16)
19 Proof
20 a) $6 = 5 \times 858 - 6 \times 714$
b) $x = -3 + 8n, y = 2 - 5n$, where $n \in \mathbb{Z}$
21 a) Proof
b) $x = 11 + 378n, y = -8 - 275n$, where $n \in \mathbb{Z}$
22 a) (i) $(1751)_8$ (ii) Proof
(iii) Follows from (i) and (ii)
b) $x \equiv 13 \pmod{45}$
23 a) (i) Proof b) (i) $x \equiv 6 \pmod 7$ (ii) 2
24 Definition and proof
25 Proof
26 a) 3
b) $x = 8 + \frac{129}{3}t = 8 + 43t, y = -20 - 108t, t \in \mathbb{Z}$
c) Proof
27 a) (i) Proof (ii) (0, 5), (2, 3), (4, 1) (mod 6)
b) Proof
28 a) Proof b) $x \equiv 18 \pmod{35}$

Chapter 3

Exercise 3.1

1 a) (i) 4 (ii) 9 (iii) {5, 6, 5, 6}
b) (i) 4 (ii) 6 (iii) {3, 3, 3, 3}
c) (i) 5 (ii) 5 (iii) {2, 1, 3, 2, 2}

2 a) No b) Yes, K_5

3 $n - 1$

4 $\frac{n(n-1)}{2}$

5 a) $v = 7$, $e = 12$ b) $v = 30$, $e = 221$
c) $v = m + n$, $e = mn$

6 8, 16

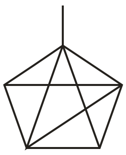
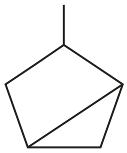
7 a) 8 b) Yes; $r = 2$, $|v| = 14$, or $r = 4$, $|v| = 7$
c) $\left\lfloor \frac{p}{2} \right\rfloor$ d) Proof

8–9 Proof

10 a, c

11 12

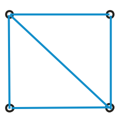
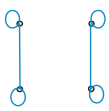
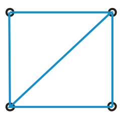
12 a) No, $|E|$ is not even.

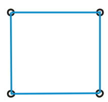
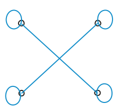
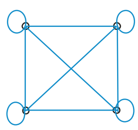
b) Yes,  c) Yes, 

Exercise 3.2

1 a) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

2 a)  b)  c) 

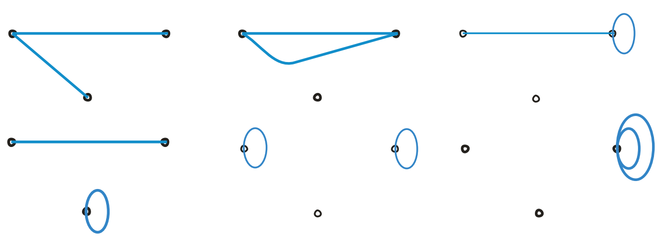
d)  e)  f) 

Graphs a) and c), and b) and e), are isomorphic.

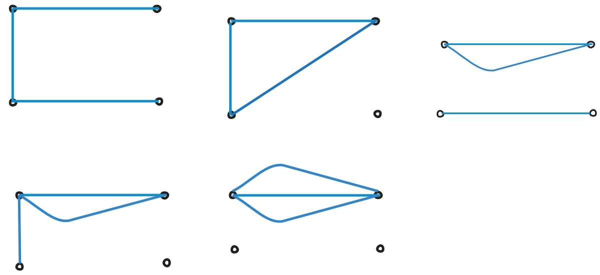
3 Isomorphic. Label the nodes, in both graphs, clockwise *a*, *b*, *c*, *d*, *e*, *f*, *g*. The correspondence *a*↔*g*, *b*↔*f*, *c*↔*e*, *d*↔*d*, *e*↔*c*, *f*↔*b*, *g*↔*a* is a homomorphism because when you rearrange the vertices in the second graph, you will have the same adjacency matrix as the first one.

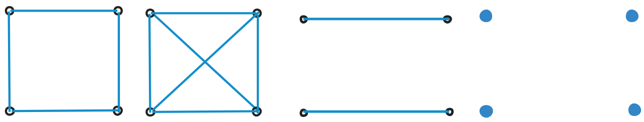
4 a) No b) No c) No d) Yes

5 2 without loops, 6 with loops



6 5 without loops, 15 with loops



7 

Exercise 3.3

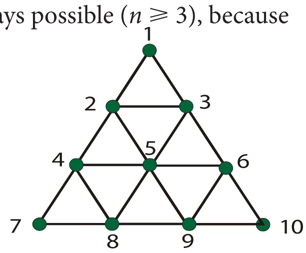
1 Vertices have even degrees.
a) 123174263456751 b) 1234543251

2 a) 1234214241 b) 12345241
c) Vertices 2 and 5 have degree 5 each.

3 a) When n is odd. b) When m and n are both even.

4 Graph 1(a) Hamiltonian: 12345671; graph 1(b) Hamiltonian: 123451.
Graph 2(a) Hamiltonian: 12341; graph 2(b) Hamiltonian path: 12345; graph 2(c) neither.

5 a) (10, 9, 6, 5, 9, 8, 5, 4, 8, 7, 4, 2, 5, 3, 2, 1, 3, 6, 10)
b) (10, 9, 8, 7, 4, 5, 2, 1, 3, 6, 10)
c) An Eulerian circuit is always possible ($n \geq 3$), because the degree of every vertex is even. A Hamiltonian cycle is also possible using the same plan as above: visit all vertices except one side, and then go back along that side.



6 Length 1 = 0; length 2 = 2; length 3 = 3, and length 4 = 10.

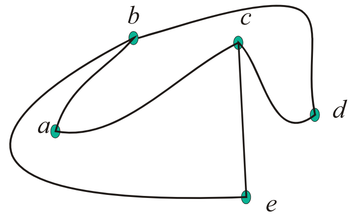
7 a) 51 between vertices not on the main diagonal, 52 for vertices on the diagonal
b) 205 between vertices not on the main diagonal, 204 for vertices on the diagonal
c) 819 between vertices not on the main diagonal, 820 for vertices on the diagonal

8 a) 48 among vertices of the 3-part, and 36 among the 4-part
b) 144 from vertices of 3-part to vertices of 4-part
c) 576 among vertices of the 3-part, and 432 among the 4-part
d) 1728 from vertices of 3-part to vertices of 4-part

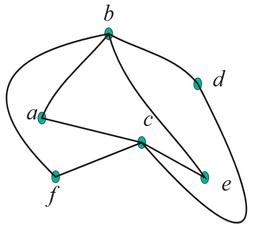
9 a) No cycle. If you start at the left, you will need to visit *c* and *d* twice. Path: *abcdef*.
b) Cycle: *abcdea*.
c) No cycle since *f* has degree 1. Path: *eabcdf*.
d) Neither cycle nor path as three vertices have degree 1.
e) No cycle, because in any of them *a* or *d* would have to be visited twice. Path: *eacdb*.
f) Cycle: *ahgfedcbia*.

Exercise 3.4

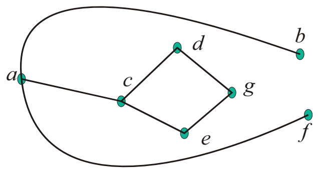
1 Planar. Redraw:



2 Planar. Redraw:



3 Planar. Redraw:



4 Not planar. *bf* and *ce* must cross, so must *ae* and *bd*.

5 15

6 15, 18

7 7, 9

8 6

9 Not planar

10 Planar

Practice questions

1 No, because there will be an edge connecting two vertices in the same component.

2 a) (i) $\binom{n}{2}$ (ii) $\binom{n}{3}$ (iii) $\binom{n}{m}$

b) $\frac{n+2}{2}$ or $\frac{n+1}{2}$

3 10

4 a) 2 b) 7

5 a) 0 b) 27

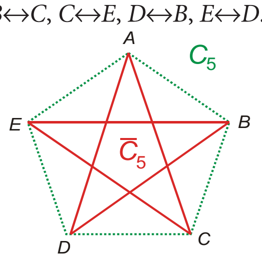
6 a) Proof b) Only C_3 is isomorphic to K_3 and W_3 to K_4.

c) Proof

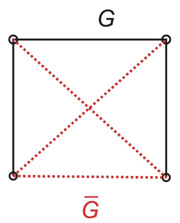
7 Proof

8 They contain odd cycles (size 3).

9 Yes; $A \leftrightarrow A$, $B \leftrightarrow C$, $C \leftrightarrow E$, $D \leftrightarrow B$, $E \leftrightarrow D$.

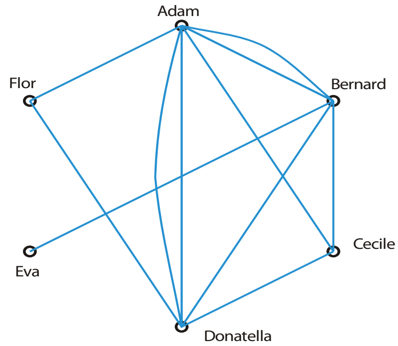


10 a) Yes:



b) No

11 a)



b) Yes, through Adam.

c) Bernard, as without him Eva is isolated.

Chapter 4

Exercise 4.1

1 a) 5, 7, 10, 11, 13, 14, 16, 17

b) 3, 1, 9

c) 3: 12, 13, 14; 7: no descendants; 15: 16, 17

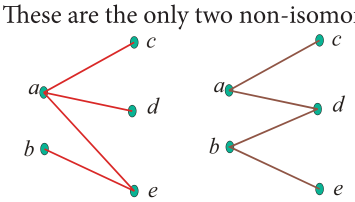
d) 4: 12; 7: no siblings; 9: no siblings

2 $|u| = 18$, $|v| = 36$, $|f| = 35$

3 31

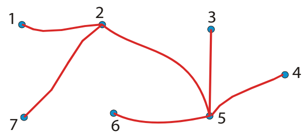
4 $\binom{n}{2}$

5 a) These are the only two non-isomorphic trees.

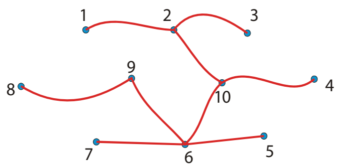


b) $\left\lfloor \frac{n+1}{2} \right\rfloor$

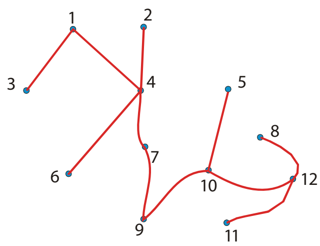
6



7



8



9 12, 23, 34, 45, 56, 67

10 12, 23, 34, 45, 56, 67, 78, 89, 9(10)

11 12, 24, 45, 58, 8(12), (12)(11), (11)9, 9(10), 47, 76, 63

12 13, 34, 45, 58, 89, 46, 67, 7(10), 12

13 17, 78, 89, 9(10), (10)(11), (11)6, 65, 54, (10)(14), 9(13), 83, 32

14 12, 23, 34, 46, 65, 5(10), (10)9, 98, 87, (10)(11), (11)(12), (12)(13), (13)(14), (14)(15), (10)(16), (16)(17), (17)(19), (19)(20), (20)(18)

15 a) 13, 12, 34, 45, 46, 67, 78, 7(10), 89

b) 12, 23, 34, 45, 56, 67, 78, 89, 7(10)

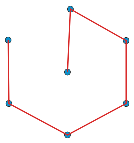
16 a) 12, 17, 7(12), 78, 83, 8(13), 89, 94, 95, 9(10), 9(14), (10)(11), (11)6

b) 12, 23, 38, 89, 94, 45, 56, 6(11), (11)(10), (10)(14), 9(13), 87, 7(12)

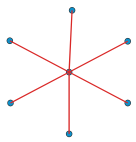
17 a) 12, 15, 23, 26, 34, 5(10), (10)7, (10)8, (10)9, (10)(11), (10)(16), (11)(12), (11)(13), (11)(14), (11)(15), (16)(17), (16)(18), (16)(20), (20)(19)

b) 12, 23, 34, 46, 65, 5(10), (10)7, 78, 89, (10)(11), (11)(12), (12)(13), (13)(14), (14)(15), (10)(16), (16)(17), (17)(18), (18)(20), (20)(19)

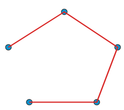
18 a) (i)



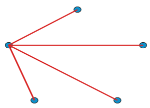
b) (i)



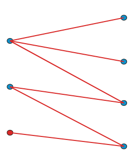
a) (ii)



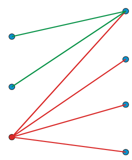
b) (ii)



a) (iii)

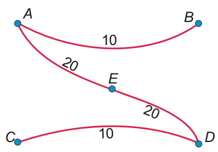


b) (iii)



Exercise 4.2

1



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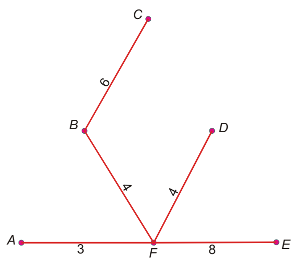
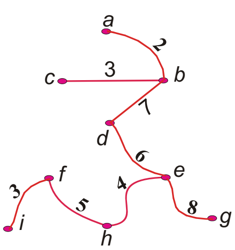
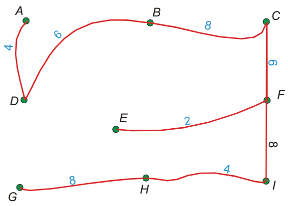


3



4



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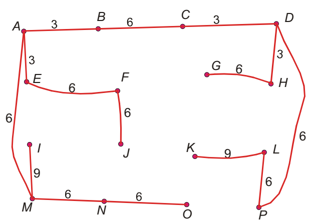

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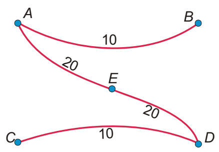

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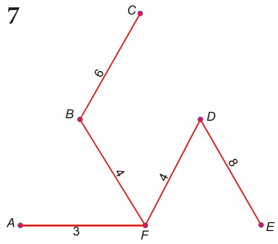

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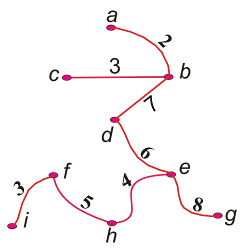
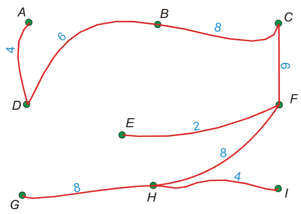


9

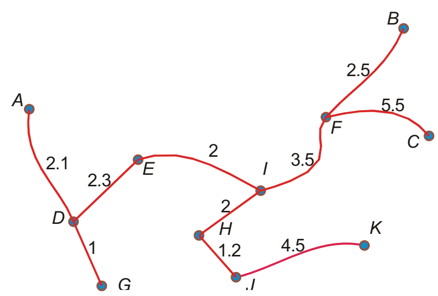


10 A few shapes are possible, one of which is similar to the answer to question 5.

11 1 and 6 have the same final tree. However, when building the tree using Kruskal's algorithm, *AB* and *CD* were added first. When using Prim's algorithm, *AB* was followed by *AE*, *ED*, and then *CD*.
With 2 and 7, there is no apparent difference. The different shapes are due to random choices.
3 and 8 have the same final tree too. Using Kruskal's algorithm, the order of addition to the tree is: *ab*, *bc*, *fi*, *he*, *fh*, *ed*, *bd*, and *eg*. Using Prim's algorithm, the order is: *ab*, *bc*, *bd*, *ed*, *he*, *fh*, *fi* and *eg*.
4 and 9 may have the same tree too. However, using Kruskal's algorithm, the order of edge addition is: *ef*, *ad*, *hi*, *cf*, *db*, *bc*, *fi*, and *gh*. Using Prim's algorithm, the order is: *ef*, *fc*, *fh*, *ih*, *cb*, *bd*, *da*, and *gh*.

5 and 10 may have the same tree too. However, using Kruskal's algorithm, the order of edge addition is: *AB*, *AE*, *CD*, *DH*, *BC*, …. Using Prim's algorithm, the order is: *AB*, *AE*, *BC*, *CD*, *DH*, ….

12



Exercise 4.3

1 70, *abedf*

2 48, *ACDEGH*

3 32, *acfimpsu*

4 *abed*

5 *A–F*: *ACDF*; *B–H*: *BCDEGH*

6 *ADBCA*, 85

7 *EDCABE* or *DEBACD*, 400

8 Vienna–Frankfurt–Prague–Moscow–Milan–Vienna: €1070.

9 New York–Paris–London–Madrid–Boston–New York: €1215.

10 *DACBED*, 550

11 *age*, 19

12 *abdfhi*, 21; *acehi*, 13

13 Without visiting any city twice: *ESYFITAPGE*, 926. Visiting *Y* twice: *EGYSYFITAPE*, 871.

14 *abcdhghcgbfgfea*, 8300

15 *abcdecjfefibjfgihgha*, 9200

Review questions

1–3 Proof

4 44

5 a) 21 b) $\binom{n}{2}$

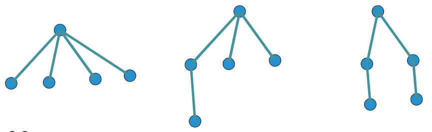
6 Proof

7 Yes; no

8 Proof

9 No; no

10

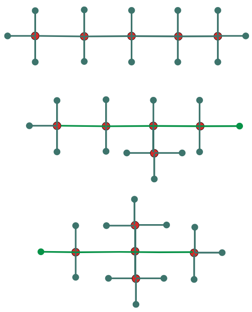


11 20

12 On the left there are 2 carbon atoms adjacent to 3 hydrogen atoms each, while on the right 3 carbon atoms have this property.

13 $2n + 2$

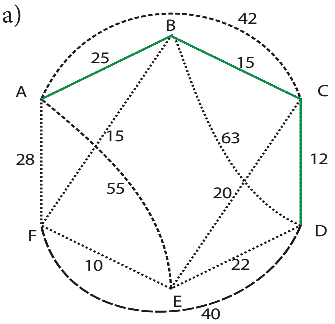
14



15 Proof

16 *BAGF*, 16

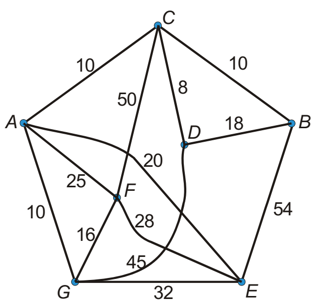
17 a)



b) ABCD, $52

18 *ACEDFGHIBA*, 8.6 km

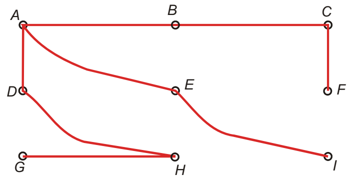
19 a)



b) Sample: *ACBDCAEFGA* with 130 000 free miles, which she can afford.

20 Yes; he will have a 20-minute break.

21 Sample for Kruskal's algorithm: *BC*, *AB*, *AE*, *CF*, *GH*, *AD*, *DH*, *EI*. Sample for Prim's algorithm: *BC*, *AB*, *AE*, *CF*, *AD*, *DH*, *GH*, *EI*. Weight = 26.



22 Sample for Kruskal's algorithm: *DG*, *HI*, *BF*, *EH*, *DE*, *FI*, *AD*, *FC*. Sample for Prim's algorithm: *DG*, *DE*, *EH*, *HI*, *IF*, *BF*, *AD*, *CF*. Weight = 45.

23 PT, SU, RU, PQ, TR, total distance of 719 km

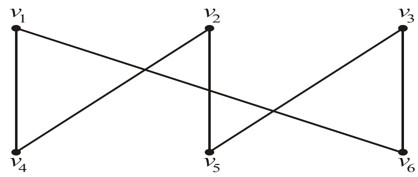
24 1043 cents (10.43 dollars)

25 35

Practice questions

1 a) Student's explanation

b)



2 No, more than two vertices have an odd degree.

3 a) Proof

b) Not isomorphic; one has a vertex of degree 4, the other does not.

4 Tree: *h*, *e*, *d*, *a*, *i*, *g*; weight = 31.

5 a)

	A	*B*	*C*	*D*	*E*	*F*
A	0	1	2	2	2	1
B	1	0	1	2	3	2
C	2	1	0	1	2	1
D	2	2	1	0	2	1
E	2	3	2	2	0	1
F	1	2	I	1	1	0

b) (i) City *F* is the most accessible since its index is 1.5. City *E* is the least accessible since its index is 10.

(ii) Cities *C* and *F* are the most accessible since each has an index of 1.5. City *E* is still the least accessible since its index is 10.

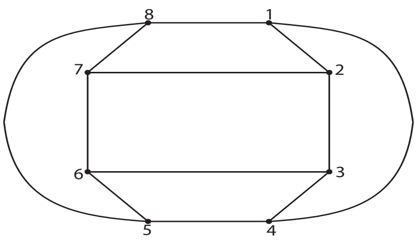
6 a)

U	1	2	3	4	5	6	7	8
1	0	1	0	1	0	0	0	1
2	1	0	1	0	0	0	1	0
3	0	1	0	1	0	1	0	0
4	1	0	1	0	1	0	0	0
5	0	0	0	1	0	1	0	1
6	0	0	1	0	1	0	1	0
7	0	1	0	0	0	1	0	1
8	1	0	0	0	1	0	1	0

b)

	A	E	B	F	C	G	D	H
A	0	1	0	1	0	0	0	1
E	1	0	1	0	0	0	1	0
B	0	1	0	1	0	1	0	0
F	1	0	1	0	1	0	0	0
C	0	0	0	1	0	1	0	1
G	0	0	1	0	1	0	1	0
D	0	1	0	0	0	1	0	1
H	1	0	0	0	1	0	1	0

c) V is planar as U is planar.



7 a)

Vertices added to the tree	Edge added	Weight
3	Ø	0
5	3, 5	10
6	3, 6	20
7	5, 7	30
10	6, 10	30
1	3, 1	40
2	1, 2	30
11	2, 11	30
9	1, 9	40
4	6, 4	40
8	7, 8	40
		310

b) Any of two paths: 1–3–4–5–6–8–10–11 or 1–3–4–5–6–9–11, with weight 80.

8 a) No; not all vertices are even.

b)

	A	B	C	D	E	U
A	0	1	0	1	0	0
B	1	0	1	1	0	1
C	0	1	0	0	1	1
D	1	1	0	0	1	1
E	0	0	1	1	0	1
U	0	1	1	1	1	0

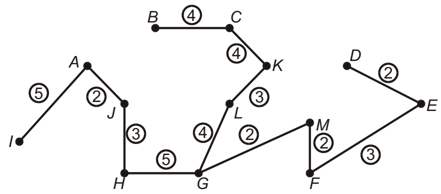
There is one walk of length 2.

c) Tree: {*AD, CE, CU, BU, AB*}, with weight of 28.

9 a) T := empty graph
for I :=1 to $n-1$
begin
  e := any edge in G with smallest weight that does not form a simple circuit when added to T
  T := T with e added
end {T is a minimum spanning tree of G.}

b) *AJ, GM, MF, DE, JH, LK, FE, GL, KC, CB, AI, HG*

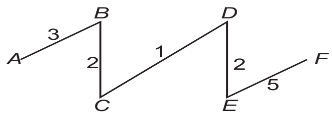
c)



Total weight = 39.

10

Iteration:	Vertices:	Labels:
First:	*A*;	*A*: 0, *B*: 3, *C*: 7, *D*: ∞, *E*: ∞, *F*: ∞
Second:	*A, B*;	*A*: 0, *B*: 3, *C*: 5, *D*: 9, *E*: ∞, *F*: ∞
Third:	*A, B, C*;	*A*: 0, *B*: 3, *C*: 5, *D*: 6, *E*: 11, *F*: ∞
Fourth:	*A, B, C, D*;	*A*: 0, *B*: 3, *C*: 5, *D*: 6, *E*: 8, *F*: 14
Fifth:	*A, B, C, D, E*;	*A*: 0, *B*: 3, *C*: 5, *D*: 6, *E*: 8, *F*: 13



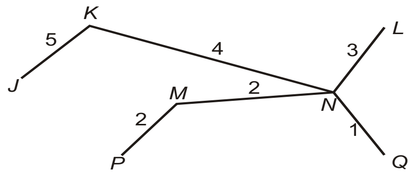
11 a) Student definition

b) Not isomorphic; *G* has a vertex of degree 3, while *H* has not.

c) *BAEBCEFCDF*

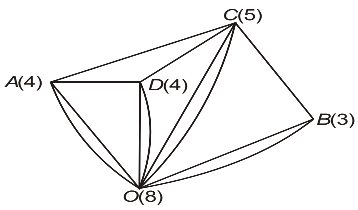
d) All vertices have even degree.

12



Weight = 17. (Other trees are also possible.)

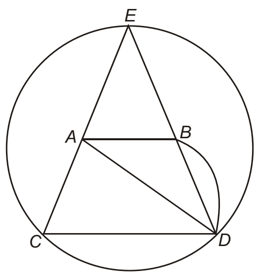
13 a)



b) Yes. The graph has exactly two vertices (*B* and *C*) with odd degree. It means that there is a path (starting at *B* or *C*) that will go once and only once through every door.

c) Yes. *O*→*D*→*A*→*C*→*B*→*O* is a Hamiltonian cycle. It means that there is a path (starting anywhere) that will go once and only once through every room before returning to its starting point.

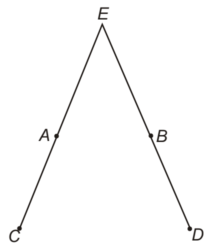
14 a)



b) The degree of every vertex is even.

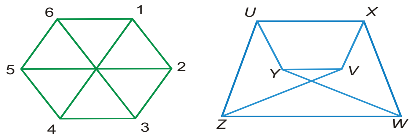
c) *AEBACDBDCEDA*

d)



15 a) Student definition b) Proof

c) (i) G is bipartite since if we label the vertices clockwise as 1, 2, 3, …, the two components will be {1, 3, 5} and {2, 4, 6}.



(ii) G and H are isomorphic: 1↔U, 2↔X, 3↔V, 4↔Y, 5↔W, 6↔Z.

(iii) No; H is bipartite, J is not.

16 a) *MQ*, *QL*, *MP*, *PN*, *NR* b) 11

17 a) G_2 does not have an Eulerian trail since four vertices have odd degrees.

b) *BABCECFEFBDEDADC*

18 a) $e = 9 \nleq 2v - 4$ b) Delete *AD*

c) Proof

19 a) 24

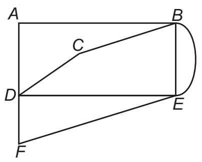
b) (i) *BDEC*

(ii) 33

c) *DBAEC* is a minimum spanning tree of weight 26. Upper bound = 26 × 2 = 52.

d) A minimum tour is 34; 33 cannot be achieved.

20 a) Example:



b) All vertices are of even order. *BEDABCDFEB* (not unique).

c) *ABCDEF*

21 23; *PQWRUST*

22 a) (i) Eulerian circuit: V_1, V_2, V_3, V_4, V_2, V_6, V_5, V_4, V_6, V_1.

(ii) Hamiltonian cycle: V_1, V_2, V_3, V_4, V_5, V_6, V_1.

b) There is no Eulerian circuit since V_2 and V_6 are now odd degree. There is a Hamiltonian cycle still, the same as above.

c) (i) Eulerian trail: V_2, V_3, V_4, V_2, V_6, V_5, V_4, V_6, V_1.

(ii) Hamiltonian path: V_2, V_3, V_4, V_5, V_6, V_1.

23 a) Every edge creates 2 degrees, with n edges there are $2n$ degrees.

b) Each vertex will have a degree of 5, 45 in total, which is not even. Hence, it is not possible.

c) See Chapter 3, page 121.

24 a)

$$A_G = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}; A_G^2 = \begin{pmatrix} 8 & 2 & 2 \\ 2 & 5 & 4 \\ 2 & 4 & 5 \end{pmatrix}; 5$$

b) Bipartite with two components: {*H*, *J*, *L*} and {*I*, *K*, *M*, *N*}.

c) Deg(U) = 3. Join *UR*. Circuit: *PQRSTRUQTUP*.

25 a) One upper bound is the length of any cycle, e.g. *ABCDEA* gives 73. Other methods also apply.

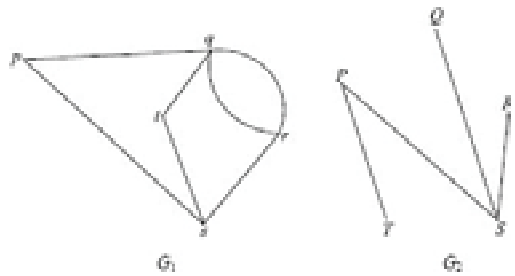
b) (i) *AB*, *AD*, *BC*, in that order.

(ii) Weight = 33; lower bound = 60

26 a) Not planar; $e = 15 \nleq 3v - 6 = 12$.

b) *BD*, *DF*, *FA*, *FE*, *EC*, in that order. Weight = 12.

27 a)



b) (i) G_1 is not simple, G_2 is simple.

(ii) Both are connected.

(iii) Both are bipartite. G_1: components are {p, r, t} and {q, s}. G_2: components are {P, R, Q} and {T, S}.

(iv) G_1 is not a tree, as it has a cycle. G_2 is a tree.

(v) G_1 contains an Eulerian trail: *rqpsrqts*. G_2 does not have an Eulerian trail since four vertices have odd degrees.

28 a) *BG*, *EF*, *ED*, *AB*, *BC*, *CF*

b) 19

29 Proof

30 a) *FD*, *FC*, *CB*, *BA*, *CE*

b) 76

31 a) (i) *D*, *E*

(ii) *EBD*

(iii) Example: *ABEFGCBDBEGDFCA*

(iv) 36

b) Example: *ABEFDGCA*

32 a) *HP*, *KQ*, *QF*, *FE*, *PB*, *ER*, *PQ*, *BC*, *CD*; 31

b) 48

c) Proof

d) Number < 1 814 400

33 a) (i) Odd degree vertices

(ii) Bipartite: components are {*B*, *D*} and {*A*, *C*, *E*}.

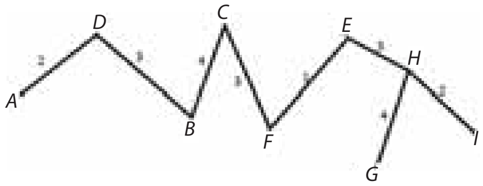
(iii)

$$G_A = \begin{pmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \end{pmatrix}; 36$$

b) 18, *PUQTRS*

34 a) *AD*, *DB*, *BC*, *CF*, *FE*, *EH*, *HI*, *HG*, *HG*

b) 22

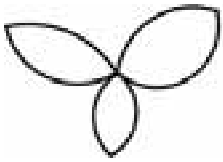


35 a) Every edge creates 2 degrees, with *e* edges there are 2*e* degrees.

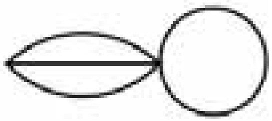
b) Student deduction

c) (i) $(n, d) = (1, 6), (2, 5), (3, 4), (5, 2)$ or $(6, 1)$

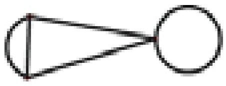
(ii) (1, 6)



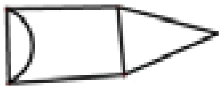
(2, 5)



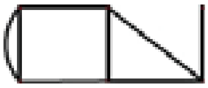
(3, 4)



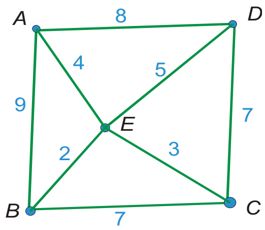
(5, 2)



(6, 1)



36 a)



b) 1

c) *ABCDEA*, weight 32; *ABCEDA*, weight 32; *ABECDA*, weight 29; *AEBCDA*, weight, 28, which is the one with the least weight.

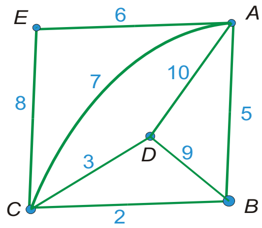
37 a) *CF*, *EF*, *BC*, *CD*, *AB*, in that order.

b) (i) $v - 1$ edges

(ii) $v - c$ edges

c) Proof

38 a) (i)



(ii) It is possible, since two vertices have odd degree.

(iii) A possible walk is *ACBDABCDCEA*; length = 55.

39 a) (i) Proof

(ii) Number of paths from v_i to v_j with a maximum length of 3.

b)–c) Proof

40 Proof

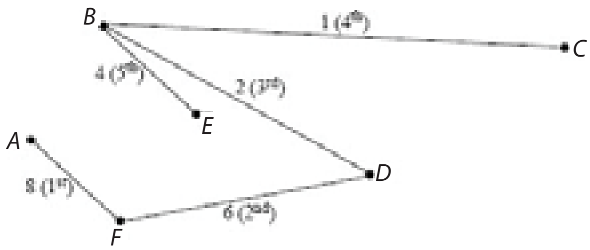
41 a) Not bipartite

b) (i)

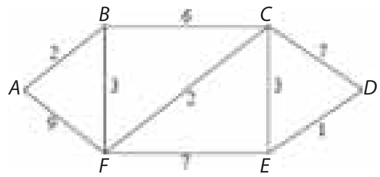
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

(ii) 13

c)



42 a)



b) *ABFCED* with length 11.